Recent developments in topological brane theories

Alessandro Tanzini

S.I.S.S.A./I.S.A.S. Trieste

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Motivations

- *topological string* simplified model for physical string: *all genus* perturbative amplitudes

- computes $F-$terms in effective actions of supersymmetric gauge theories: world–sheet instantons in A topological string $\leftrightarrow$ instantons in $\mathcal{N}=2$ gauge theories

- non-perturbative effects in topological string:

  - topological M theory as unifying description of topological A and B models

  - membrane instantons and $\mathcal{N}=1$ superpotentials

  - black hole entropy

Plan

- S–duality in topological string and topological M theory
- topological action for membranes on manifolds with G_2 structure and relation with A and B models
- observables and moduli spaces
- topological p–branes on special manifolds
- membrane theory on Seifert manifolds

Based on:

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Other references:

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Topological string - A model

maps

$$X: \quad \Sigma_g \longmapsto P$$

For topological strings P is a Calabi-Yau three-fold. Given the Kähler two-form ω on P, define

$$\int_{\Sigma_g} X^*(\omega) = d$$

$d \in \mathbf{Z}$: degree of the map, topological invariant.

String action functional:

$$S_{string} = t \int_{\Sigma_g} d\mathrm{vol}(\Sigma_g) \geq \frac{t}{2} \int_{\Sigma_g} X^*(\omega)$$

calibration condition.

minima: $\partial X^I = 0 \quad \bar{\partial} X^{\bar{I}} = 0$ holomorphic maps

$$\Big\{\text{minima of } \; S_{string}\Big\} = \coprod_{d=0}^{\infty} M_{g,n}(P,d)$$

$M_{g,n}(P,d)$ moduli space of holomorphic maps of degree d (with n marked points)

genus zero free energy:

$$\mathcal{F}_0 = \sum_d \mathcal{N}_{0,d} Q^d \quad Q^d \equiv \mathrm{e}^{-dt}$$

$\mathcal{N}_{0,d}$ is a *Gromov-Witten invariant*: "number" of genus zero holomorphic curves with degree d in P.

S–duality in topological string

A–model:

closed strings amplitudes $\rightarrow$ Kähler data

open strings: consistency of $B.C.$ $\Rightarrow$ A–branes wrap Lagrangian submanifolds γ, with Vol$(\gamma) = \Omega|_\gamma$, Ω holomorphic three–form

e.g.: deformed conifold $x^2 + y^2 + w^2 + z^2 = \mu$ on C^4, $\mu \in R^+$ is T^*S^3 (phase space of a particle on a 3–sphere). Lagrangian submanifold is S^3 (configuration space of the particle) $\Rightarrow$ Chern–Simons theory on S^3.

$\Rightarrow$ *non–perturbative* amplitudes: complex structure data

B–model:

closed strings amplitudes $\rightarrow$ complex data

open strings: holomorphic submanifolds couple to Kähler data

e.g.: D5 brane wrapping the Calabi–Yau support a $U(1)$ *holomorphic Chern–Simons* theory. Non–perturbative effects from D1 and D(-1) branes couple to Kähler form ω:

$$\exp\left[-\int \omega \wedge F \wedge F - \frac{1}{g_B}\int F \wedge F \wedge F\right]$$

Donaldson-Thomas invariants

suggestive picture:

A–model: *perturbative* Kähler *non–perturbative* complex

B–model: *perturbative* complex *non–perturbative* Kähler

examples/indications:

- topological strings on twistorial Calabi–Yau [Neitzke and Vafa, 2004]

- D-brane instanton amplitudes in B model ↔ Kähler gravity on toric Calabi–Yau's [Iqbal, Nekrasov, Okounkov and Vafa, 2003, Nekrasov Ooguri and Vafa, 2004]

- duality Donaldson-Thomas invariants ↔ Gromov-Witten invariants [Maulik, Nekrasov, Okounkov and Pandharipande 2003]; holomorphic BF theory [Baulieu and A.T., 2004]

⇒ S–duality conjecture between A and B models *on the same* Calabi–Yau manifold.

The complete theory must contain both the *Kähler* and *complex structure* data of the manifold.

Topological M theory

basic idea: going one dimension up as in physical M theory

e.g. take $M = CY \times S^1$: this is a manifold with G_2–holonomy:

$$\Phi = Re\,\Omega + dX^7 \wedge \omega$$

three–form invariant under G_2 rotations of a frame, with

$$d\Phi = 0 \qquad\qquad d * \Phi = 0$$

these equations can be obtained as the extrema of the *Hitchin functional*

$$V_7(\Phi) = \int \Phi \wedge *_\Phi \Phi$$

assuming that Φ is closed and vary in a fixed cohomology class $\Phi = \Phi_0 + dB$.

$\Rightarrow$ *Hitchin functional* as effective action for topological M theory.

[R. Dijkgraaf, S. Gukov, A. Neitzke and C. Vafa, 2004]

OBS.

- the metric (and hence the Hodge star) is derived from Φ

- the functional above is well defined on manifolds with G_2–structure and simply gives the volume of that manifolds!

- Φ naturally couples with a *membrane* world–volume

$$S_{top} = \int_{\Sigma_3} X^*(\Phi),$$

where

$$X : \Sigma_3 \to M$$

$\Rightarrow$ membrane theory

Membranes on G_2 manifolds

physical (Nambu–Goto) Euclidean membrane action

$$S = \int d^3\sigma \, \sqrt{\gamma} = \int d^3\sigma \, \sqrt{\det(\partial_\alpha X^\mu g_{\mu\nu} \partial_\beta X^\nu)}$$

auxiliary world–volume metric $h_{\alpha\beta}$

$$S = \frac{1}{2} \int d^3\sigma \, \sqrt{h}(h^{\alpha\beta} \partial_\alpha X^\mu g_{\mu\nu} \partial_\beta X^\nu - 1)$$

in the gauge: $h_{00} = \det(\mathrm{h_{ab}})$, $\mathrm{h_{0a}} = 0$

$$S = \frac{1}{2} \int d^3\sigma \, \left(\dot{X}^\mu g_{\mu\nu} \dot{X}^\nu + \det(\partial_a X^\mu g_{\mu\nu} \partial_b X^\nu) \right)$$

global $\Sigma_g \times S^1$ topology.

minima of the membrane action

bound

$$\int d^3\sigma \; \left(\dot{X}^\mu \pm \Phi^\mu{}_{\nu\rho}\partial_1 X^\nu \partial_2 X^\rho\right) g_{\mu\lambda} \left(\dot{X}^\lambda \pm \Phi^\lambda{}_{\sigma\tau}\partial_1 X^\sigma \partial_2 X^\tau\right) \geq 0$$

on manifolds with G_2 structure, one has a generalisation of the three-dimensional vector product

$$(\partial_1 X \times \partial_2 X)^\mu \equiv \Phi^\mu{}_{\nu\rho}\partial_1 X^\nu \partial_2 X^\rho$$

such that

$$(\partial_1 X \times \partial_2 X) \cdot (\partial_1 X \times \partial_2 X) = \det(\partial_a X \cdot \partial_b X)$$

then

$$\frac{1}{2}\int d^3\sigma \; \left(\dot{X}^\mu g_{\mu\nu} \dot{X}^\nu + \det(\partial_a X^\mu g_{\mu\nu} \partial_b X^\nu)\right) \geq \mp\frac{1}{6}\int X^*(\Phi),$$

membrane instantons

$$\dot{X}^{\mu} \pm \Phi^{\mu}{}_{\nu\rho}\partial_1 X^{\nu}\partial_2 X^{\rho} = 0$$

this condition is equivalent to the calibration condition

$$d\,vol(\Sigma_3) = \mp\frac{1}{6}\,X^*(\Phi)$$

$\Rightarrow$ for manifolds with G_2 holonomy ($d\Phi = d * \Phi = 0$), membrane instantons are *associative submanifolds* calibrated by Φ.

Membrane instantons are characterised at the classical level by the topological invariant

$$\frac{1}{6}\int_{\Sigma_3} X^*(\Phi) = 2\pi n, \qquad n \in Z$$

on $M = CY \times S^1$, $\Phi = Re\,\Omega + dX^7 \wedge \omega \Rightarrow$

- membranes wrapping S^1: $X^7 = k\sigma^2 \Rightarrow$

$$\partial_0 X^M + J^M{}_N \partial_1 X^N = 0$$

holomorphic curves in the $CY \Rightarrow$ A-model

- membranes localised in S^1: $X^7 = \text{const.} \Rightarrow$

$$\partial_0 X^M + Re(\Omega)^M{}_{NP} \partial_1 X^N \partial_2 X^P = 0$$

special lagrangian cycles of the $CY \Rightarrow$ B-model

inclusion of flux:

our topological model can be considered also in presence of a non–trival flux

$$H = d\Phi$$

In this case one has to start with the classical action

$$S_{top} = \int_{\Sigma_3} X^*(\Phi) - \int_{\Sigma_4} X^*(H),$$

with $\Sigma_4 = \partial\Sigma_3$, which is invariant due to the identity

$$\delta \int_{\Sigma_n} X^*(\Phi) = \int_{\partial\Sigma_n} X^*(i_{\delta X}\Phi) + \int_{\Sigma_n} X^*(i_{\delta X} d\Phi)$$

The reduction of this model on $M = X \times S^1$ should describe topological sigma models on a manifold X with *generalised complex structure*.

Quantization of the membrane action

classical action

$$S = -\frac{1}{6}\int_{\Sigma_3} X^*(\Phi)$$

$d\Phi = 0 \Rightarrow$ invariant under shifts $\delta X^\mu = \epsilon^\mu$, with BRST

$$sX^\mu = \psi^\mu, \qquad s\psi^\mu = 0, \qquad s\bar{\psi}^\mu = b^\mu, \qquad sb^\mu = 0$$

gauge–fixing

$$\mathcal{F}^\mu = \dot{X}^\mu + \Phi^\mu{}_{\nu\rho}\partial_1 X^\nu \partial_2 X^\rho + \frac{1}{2}\Gamma^\mu{}_{\sigma\rho}\bar{\psi}^\sigma\psi^\rho$$

term quadratic in the ghosts required for manifest general co-variance (as in topological quantum mechanics and topological sigma model) [Baulieu and Singer, 1989] gauge–fixed action

$$S_{GF} = -\tfrac{1}{6}\int X^*(\Phi)$$

$$+\int d^3\sigma\ s\left(\bar\psi^\mu(g_{\mu\nu}\dot X^\nu + \Phi_{\mu\nu\rho}\partial_1 X^\nu \partial_2 X^\rho + \frac{1}{2}\Gamma_{\mu\sigma\rho}\bar\psi^\sigma\psi^\rho - \frac{1}{2}g_{\mu\nu}b^\nu)\right)$$

by integrating b^μ

$$S_{GF} = \int d^3\sigma\left(\frac{1}{2}\dot X^\mu g_{\mu\nu}\dot X^\nu + \frac{1}{2}\mathrm{det}(\partial_a X^\mu g_{\mu\nu}\partial_b X^\nu)\right.$$

$$\left.-\bar\psi^\mu g_{\mu\nu}\nabla_0\psi^\nu - \Phi_{\mu\nu\rho}\bar\psi^\mu\nabla_a\psi^\nu\partial_b X^\rho\epsilon^{ab} + \frac{1}{4}\mathcal{R}_{\mu\sigma\lambda\rho}\bar\psi^\mu\psi^\rho\bar\psi^\sigma\psi^\lambda\right)$$

Partition function and moduli space

The membrane theory has a non-trivial partition function Z: in fact the fermionic zero-modes of D and $D^\dagger$ can be soaked up by the $\mathcal{R}\psi^2\bar{\psi}^2$ term of the membrane action

Z compute the Euler characteristic of the moduli space of associative maps (similar to topological quantum mechanics and twisted $\mathcal{N} = 4$ SYM theory).

$\Rightarrow$ same as the partition function of super-membrane theory studied in [Beasley and Witten, 2003] for M–theory compactifications on G_2 manifolds.

moduli space $\mathcal{M}$ for associative cycles:

deformations of associative submanifolds studied by McLean [McLean, 1998]

Our field ψ is a section of $T\mathcal{M}$. Choose the static gauge

$$X^\alpha = \sigma^\alpha \,,$$

$$\dot{X}^i \pm \Phi^i{}_{aj}\epsilon^{ab}\partial_b X^j = 0 \,,$$

where $\alpha = 0, 1, 2$ and $i = 3, \ldots, 6$.

BRST variation $\Rightarrow$ McLean twisted Dirac operator.

In this gauge our model reproduces the results of supermembrane theory.

Topological p-branes

p-brane action:

$$S = \frac{1}{2}\int d^{p+1}\sigma\ \left(\dot{X}^\mu g_{\mu\nu}\dot{X}^\nu + \mathrm{det}(\partial_a X^\mu g_{\mu\nu}\partial_b X^\nu)\right),$$

assuming a (p+1)-form on target space M, write the bound

$$\left(\dot{X}^\mu \pm \phi^\mu{}_{\nu_1\ldots\nu_p}\partial_1 X^{\nu_1}\ldots\partial_p X^{\nu_p}\right) g_{\mu\lambda}\left(\dot{X}^\lambda \pm \phi^\lambda{}_{\sigma_1\ldots\sigma_p}\partial_1 X^{\sigma_1}\ldots\partial_p X^{\sigma_p}\right) \geq 0.$$

If ϕ and g correspond to a vector cross product structure on M

$$\frac{1}{2}\int d^{p+1}\sigma\ \left(\dot{X}^\mu g_{\mu\nu}\dot{X}^\nu + \mathrm{det}(\partial_a X^\mu g_{\mu\nu}\partial_b X^\nu)\right) \geq \mp\frac{1}{(p+1)!}\int X^*(\phi).$$

If $d\phi = 0$ r.h.s. is a *topological term*. $\Rightarrow$ gauge-fixed action as in the membrane case

$$S_{GF} = -\frac{1}{(p+1)!}\int X^*(\phi)$$

$$+\frac{1}{2}\int d^{p+1}\sigma \; s\left(\bar{\psi}^\mu (g_{\mu\nu}\dot{X}^\nu + \phi_{\mu\nu_1\ldots\nu_p}\partial_1 X^{\nu_1}\ldots\partial_p X^{\nu_p})\right)$$

localises on

$$\dot{X}^\mu + \phi^\mu{}_{\nu_1\ldots\nu_p}\partial_1 X^{\nu_1}\ldots\partial_p X^{\nu_p} = 0,$$

submanifolds of M calibrated by ϕ.

classification of vector cross products: [Brown and Gray]

(1) $p = d - 1$ and ϕ is the volume form of the manifold.

(2) d even, $p = 1$, one-fold cross product $J : TM \to TM$, $J^2 = -1$ almost complex structure. Associated 2-form: Kähler form $\Rightarrow$ topological A sigma model.

(3) 2-fold cross product ($p = 2$) on a 7-manifold, G_2-structure $\Phi \Rightarrow$ topological membrane.

(4) 3-fold cross product ($p = 3$) on an 8-manifold: $Spin(7)$-structure with (Cayley) 4-form $\Psi \Rightarrow$ topological F theory

$$CY_3 \times T^2 \quad \Longrightarrow \quad CY_3 \times S^1 \quad \Longrightarrow \quad CY_3,$$

where $CY_3 \times T^2$ is a $Spin(7)$-manifold.

Hamiltonian formalism

momenta:

$$p_\mu = g_{\mu\nu}\dot{X}^\nu - \Gamma_{\lambda\mu\nu}\bar{\psi}^\lambda\psi^\nu, \qquad p_{\psi\mu} = g_{\mu\nu}\bar{\psi}^\nu,$$

Poisson brackets:

$$\{X^\mu(\sigma), p_\nu(\sigma')\} = \{\psi^\mu(\sigma), \bar{\psi}_\nu(\sigma')\}_+ = \delta^\mu{}_\nu\delta^2(\sigma - \sigma')$$

BRST in phase space

$$Q = \int d^2\sigma\ \psi^\mu(p_\mu + \Phi_{\mu\nu\rho}\partial_1 X^\nu \partial_2 X^\rho),$$

anti-BRST

$$\bar{Q} = \int d^2\sigma\ \bar{\psi}_\tau g^{\tau\mu}(p_\mu - \Phi_{\mu\nu\rho}\partial_1 X^\nu \partial_2 X^\rho + \Gamma_{\sigma\mu\rho}\bar{\psi}^\sigma\psi^\rho).$$

<u>REMARK</u>: the choice of the bilinear fermionic term in the gauge function is the only one for which

$$\bar{Q} = -Q^\dagger$$

.

gauge-fixed Hamiltonian

$$H_{GF} = \frac{1}{2}\{Q, \bar{Q}\}_+, \qquad Q^2 = 0, \qquad \bar{Q}^2 = 0.$$

Membranes on Seifert manifolds

Seifert manifold $M_{g,p}$: total space of the $U(1)$ bundle w. charge p on a Riemann surface w. genus g, *e.g.*:

- $p = 0, g$: $\quad M_{g,0} = \Sigma_g \times S^1$
- $p = 1, g = 0$: $\quad M_{0,1} = S^3 \quad$ ($U(1)$-bundle $\Leftrightarrow$ Hopf map)
- $p, g = 0$: $\quad M_{0,p} = S^3/\mathbf{Z_p}$ Lens spaces

Globally defined one-form κ dual to the fundamental vector field n generating $U(1)$ action: $\quad \imath_n \kappa = 1$

Curvature: $\quad d\kappa = p\pi^*(\omega) \quad , \quad \imath_n d\kappa = 0$

ω symplectic form on Σ_g.

equation for associative maps:

$$d\kappa(dX^\mu + \Phi^\mu_{\nu\rho} * dX^\nu dX^\rho) = 0$$

Hodge $*$ w.r.t. natural metric on Seifert manifolds:

$$ds^2 = h_S d\sigma \otimes d\sigma = ds^2_\Sigma + \kappa \otimes \kappa$$

then $\qquad d\kappa \; dX^\mu = p * k \; dX^\mu = \sqrt{h_S} \;\; \imath_n dX^\mu \; d^3\sigma.$

The square of this gauge fixing cd gives rise to the physical Nambu-Goto action for the membrane in the gauge $h = h_S$

momentum of the Nambu-Goto action:

$$p_\mu = \frac{\delta S_{NG}}{\delta(\imath_n dX^\mu)} = \sqrt{\gamma} \; \kappa_\alpha \gamma^{\alpha\beta} g_{\mu\nu} \partial_\beta X^\nu$$

Perspectives

The membrane theory we presented plays the analogous rôle for topological M theory as the topological sigma model for topological strings.

Complete theory involves *sum over world-volume metrics* $h_{\alpha\beta} \Rightarrow$ coupling to 3D topological gravity

On Seifert manifolds this can be done! Gravity $\Rightarrow$ moduli space as a fibration

$$\mathbf{T}^{\mathbf{g}} \longrightarrow \widehat{\mathcal{M}}_g \longrightarrow \mathcal{M}_g$$

$\mathcal{M}_g$ moduli space of the Riemann surface Σ_g. $\mathbf{T}^{\mathbf{g}}$ moduli space of flat connections on Σ_g.

interesting issues:

- dualities in topological M theory and black hole physics: Donaldson-Thomas vs Gopakumar-Vafa [Gaiotto, Strominger, Yin and Dijkgraaf, Vafa, Verlinde, 2006]

- higher dimensional generalization of Gromov-Witten invariants - Joyce invariants for Lagrangian submanifolds

- super Calabi-Yau's as target manifolds

- topological F theory

Observables

For a non trivial element $[K] \in H^q(M)$ formally define the cocycles

$$\mathcal{C}_3^{q-3} = \frac{1}{6} K_{\mu_1 \ldots \mu_q} dX^{\mu_1} \wedge dX^{\mu_2} \wedge dX^{\mu_3} \psi^{\mu_4} \ldots \psi^{\mu_q},$$

$$\mathcal{C}_2^{q-2} = \frac{1}{2} K_{\mu_1 \ldots \mu_q} dX^{\mu_1} \wedge dX^{\mu_2} \psi^{\mu_3} \psi^{\mu_4} \ldots \psi^{\mu_q},$$

$$\mathcal{C}_1^{q-1} = K_{\mu_1 \ldots \mu_q} dX^{\mu_1} \psi^{\mu_2} \psi^{\mu_3} \psi^{\mu_4} \ldots \psi^{\mu_q},$$

$$\mathcal{C}_0^{q} = K_{\mu_1 \ldots \mu_q} \psi^{\mu_1} \psi^{\mu_2} \psi^{\mu_3} \psi^{\mu_4} \ldots \psi^{\mu_q},$$

in $\mathcal{C}_i^{q-i}$ the upper index stands for the *ghost number* and the lower index for the *degree* of the differential form on Σ_3.

Using the BRST transformations we can derive the descent equations

$$s\mathcal{C}_3^{q-3} = \frac{1}{q-2} d\mathcal{C}_2^{q-2}, \qquad s\mathcal{C}_2^{q-2} = \frac{1}{q-1} d\mathcal{C}_1^{q-1},$$

$$s\mathcal{C}_1^{q-1} = \frac{1}{q} d\mathcal{C}_0^{q}, \qquad s\mathcal{C}_0^{q} = 0.$$

$\mathcal{C}_0^q$: BRST-invariant *local* observables labeled by the elements of the de Rham complex $H^\bullet(M)$.

$\mathcal{C}_i^{q-i}$: BRST-invariant *non-local* observables as integrals

$$\int_{c_i} \mathcal{C}_i^{q-i},$$

c_i: i-cycle on Σ_3.

The membrane action displays a $U(1)$ ghost number symmetry with $(X, \psi, \bar{\psi})$ having charges $(0, 1, -1)$ respectively.

The operators C_0^q , $q \neq 0$ have a *non-vanishing* ghost number $\Rightarrow$ in order to have non-vanishing correlators there should be a *ghost number anomaly*

linearised fermion equations

$$D\psi^\mu = \nabla_0 \psi^\mu + \Phi^\mu{}_{\nu\rho} \epsilon^{ab} \nabla_a \psi^\nu \partial_b X^\rho = 0,$$

$$D^\dagger \bar{\psi}^\mu = \nabla_0 \bar{\psi}^\mu - \Phi^\mu{}_{\nu\rho} \epsilon^{ab} \nabla_a \bar{\psi}^\nu \partial_b X^\rho = 0.$$

the operator $D^\dagger$ is the adjoint of $D \Rightarrow$
ghost number anomaly $\equiv ind(D)$.
$ind(D) = 0$ by index theorem $\rightarrow$ no ghost number anomaly

The only non-vanishing correlators are C_i^0 , $i = 1, 2, 3$, where C_3^0 corresponds to our classical action and its variations in $H^3(M)$.