Local Index Formulas on Quantum Spheres

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Introduction

- Archetype of a space described by a noncommutative algebra: phase-space of quantum mechanics, $[x,p] = i\hbar$.
  Geometric concepts seem to lose their meaning.
- Aim of noncommutative geometry (NCG for short): to translate (differential) geometric properties into algebraic ones, that can be studied with algebraic tools and possibly generalized to noncommutative algebras.

Geometry	"is dual to"	**Algebra**
Compact topological spaces X		Unital commutative C^*-algebras $C(X)$
Compact smooth manifolds M		Comm. Frechét pre-C^*-algebras $C^\infty(M)$
Vector bundles E over X		Finite projective $C(X)$-modules, $\Gamma(E)$
Smooth vector bundles E over M		Finite projective $C^\infty(M)$-modules, $\Gamma^\infty(E)$

Definition of Podleś spheres

On the interphase between q-groups and NCG.

$\mathcal{A}(S^2_{qs})$ is the $*$-algebra generated by $A = A^*$, B and B^* with relations:

$$AB = q^2 BA \,, \quad BB^* + (A+s^2)(A-1) = 0 \,, \quad B^*B + (q^2A+s^2)(q^2A-1) = 0 \,.$$

$]0,1] \ni q =$ deformation parameter, $s \in [0,1]$ an additional parameter.

$q = 1 \Rightarrow 2$-sphere with center and radius depending on s.

$q \neq 1 \Rightarrow$ the algebra is noncommutative, we call Podleś sphere the underlying ‘virtual space’.

Some references:

$C(S^2_{qs}) \longrightarrow$ Podleś, Lett. Math. Phys. **14** (1987) 521.

$C^\infty(S^2_{qs}) \longrightarrow$ D’Andrea and Dąbrowski, Lett. Math. Phys. **75** (2006) 235.

q-monopoles $\longrightarrow$ Brzeziński and Majid, Commun. Math. Phys. **213** (2000) 491.

Equivariant reps. $\longrightarrow$ Schmuedgen and Wagner, Preprint math.QA/0305309.

Basic q-monopole and chiral representations

Deformation of the Bott projector:

$$p_{qs} := \frac{1}{1+s^2}\begin{pmatrix} s^2+A & B \\ B^* & 1-q^2A \end{pmatrix} .$$

Classically ($q=1$) it describes a charge 1 magnetic monopole.

We define the Hilbert spaces

$$\mathcal{H}_+ := L^2(S^2_{qs})^2 p_{qs} \ , \qquad \mathcal{H}_- := L^2(S^2_{qs})^2(1-p_{qs}) \ .$$

($q=1 \Rightarrow \mathcal{H}_\pm = L^2$ Weyl spinors).

The representation of $\mathcal{A}(S^2_{qs})$ on $\mathcal{H}_\pm$ can be explicitly computed in a basis of harmonic spinors $|l,m\rangle_\pm$, $l \in \mathbb{N}+\frac{1}{2}$ and $m=-l,-l+1,\dots,l$.

Just to give an idea:

$$\begin{aligned}
A\,|l,m\rangle_+ = &-q^{m-1}\sqrt{\frac{[l+\frac{1}{2}][l+\frac{3}{2}]}{[2l+1][2l+3]}}\,\frac{\sqrt{[l+m+1][l-m+1]}}{[2l+2]}\times \\
&\times\sqrt{(q^{l+\frac{1}{2}}+q^{-l-\frac{1}{2}}s^2)(q^{l+\frac{3}{2}}s^2+q^{-l-\frac{3}{2}})}\,|l+1,m\rangle_+ \\
&-q^{m-1}\sqrt{\frac{[l-\frac{1}{2}][l+\frac{1}{2}]}{[2l-1][2l+1]}}\,\frac{\sqrt{[l+m][l-m]}}{[2l]}\times \\
&\times\sqrt{(q^{l+\frac{1}{2}}+q^{-l-\frac{1}{2}}s^2)(q^{l-\frac{1}{2}}s^2+q^{-l+\frac{1}{2}})}\,|l-1,m\rangle_+ \\
&+\left((1-s^2)\frac{(1-q^2)[l-\frac{1}{2}][l+\frac{3}{2}]+1}{1+q^2}-1\right)\times \\
&\times\frac{[l+m+1][l-m]-q^{-2}[l+m][l-m+1]}{[2l][2l+2]}\,|l,m\rangle_+ \\
&+\frac{1-s^2}{1+q^2}\,|l,m\rangle_+ \quad ,
\end{aligned}$$

plus similar formulas for $A\,|l,m\rangle_-$ and $B\,|l,m\rangle_\pm$.

Here $[x] := (q^x-q^{-x})/(q-q^{-1})$ is the q**-analogue** of $x\in\mathbb{C}$.

A primer on spectral triples

Definition of **spectral triple** $(\mathcal{A}, \mathcal{H}, D)$:

- $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a $*$-algebra with 1, $\mathcal{H}$ a (separable) Hilbert space;
- D is a selfadjoint operator on (a dense subspace of) $\mathcal{H}$, $(D^2+1)^{-1/2} \in \mathcal{K}(\mathcal{H})$ and $[D, a] \in \mathcal{B}(\mathcal{H})\ \forall\ a \in \mathcal{A}$;

⇒ **dimension axiom**: "$\exists\ d \in \mathbb{R}^+$ s.t. the eigenvalues of $|D|^d$ (in increasing order, counting multiplicities) diverge linearly";

⇒ the triple is **even** if $\exists\ \gamma = \gamma^*$, such that $\gamma^2 = 1$, $\gamma D + D\gamma = 0$ and $a\gamma = \gamma a\ \forall\ a \in \mathcal{A}$.

Prototype: $(C^\infty(M), L^2(M, \mathcal{S}), \not{D})$. $d = \dim M$. $\exists\ \gamma$ iff d is even.

Let $d = 2$. A **real structure** is an odd antilinear isometry J on $\mathcal{H}$ s.t.:

$$J^2 = -1\ , \quad [D, J] = 0\ , \quad [a, JbJ^{-1}] = 0\ , \quad [[D, a], JbJ^{-1}] = 0\ , \quad \forall\ a, b \in \mathcal{A}\ .$$

Analytic properties

$(\mathcal{A},\mathcal{H},D)$ is called **regular** if

$$\mathcal{A} \cup [D,\mathcal{A}] \subset \bigcap_{j\in\mathbb{N}} \operatorname{dom}\delta^j \quad , \qquad \delta(\,.\,) = [|D|,\,.\,] \quad .$$

To each $a \in \mathfrak{B} := \bigcup_{j\in\mathbb{N}} \delta^j(\mathcal{A}\cup[D,\mathcal{A}])$ we associate a ζ-function:

$$\zeta_a(z) := \mathrm{Trace}_{\mathcal{H}}(a|D|^{-z}) \ ,$$

holomorphic for $z \in \mathbb{C}$ with $\mathrm{Re}\, z$ sufficiently large.

Definition

*A spectral triple has **dimension spectrum** Σ iff $\Sigma \subset \mathbb{C}$ is a countable set and for all $a \in \mathfrak{B}$, $\zeta_a(z)$ extends to a meromorphic function on $\mathbb{C}$ with poles in Σ as unique singularities.*

Residues of ζ-func. $\leftrightarrow$ pairing $K_\bullet(\mathcal{A}) \otimes PHC^\bullet(\mathcal{A}) \to \mathbb{Z}$ (LIF).

Dirac operators on Podleś spheres

We[1] constructed a family of $SU_q(2)$-equivariant spectral triples on S^2_{qs}.

Peculiar example: $\mathcal{H} := \mathcal{H}_+ \oplus \mathcal{H}_-$ discussed before, with natural grading, Dirac operator and 'quasi' real structure defined by:

$$D\,|l,m\rangle_\pm := (l+\tfrac{1}{2})\,|l,m\rangle_\mp \quad , \qquad J\,|l,m\rangle_\pm := (-1)^{m+1/2}\,|l,-m\rangle_\mp \quad .$$

Proposition

$(\mathcal{A}(S^2_{qs}),\mathcal{H},D)$ is a 2^+-summable regular even spectral triple.

Computations are simplified by neglecting smoothing operators

$$\mathrm{OP}^{-\infty} := \{T\in\mathcal{B}(\mathcal{H})\,|\,\langle l,m|T|l',m'\rangle \text{ is a rapid decay matrix}\} \quad .$$

$\zeta_T(z)$ holomorphic on $\mathbb{C}$ $\forall\, T\in\mathrm{OP}^{-\infty}$ $\Rightarrow$ $\mathrm{OP}^{-\infty}$ do not contribute to singularities of ζ-functions.

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[1] Joint work with L. Dabrowski, G. Landi and E. Wagner.

'Approximate' representation

Let $\hat{\mathcal{H}}$ be an Hilbert space with orthonormal basis $|l,m\rangle_\pm$, $l,m \in \mathbb{Z}+\frac{1}{2}$. Think to $\mathcal{H} \subset \hat{\mathcal{H}}$ as defined by $l > 0$ and $|m| \leq l$. A rep. $\pi : \mathcal{A}(S^2_{qs}) \to \mathcal{B}(\hat{\mathcal{H}})$ is:

$$\begin{aligned}\pi(A)\,|l,m\rangle_\pm = &-s\,q^{l+m-1}\sqrt{1-q^{2(l+m)}}\,|l-1,m\rangle_\pm \\ &+(1-s^2)q^{2(l+m)}\,|l,m\rangle_\pm \\ &-s\,q^{l+m}\sqrt{1-q^{2(l+m+1)}}\,|l+1,m\rangle_\pm \quad , \\ \pi(B)\,|l,m\rangle_\pm = &\ \dots\end{aligned}$$

Let $\tilde{\pi}$ be the projection onto $\mathcal{H}$. $\Rightarrow \tilde{\pi}$ much simpler than the original rep.

Lemma $a - \tilde{\pi}(a) \in \mathrm{OP}^{-\infty}$ for all $a \in \mathcal{A}(S^2_{qs})$.

Proposition

The dimension spectrum is $\Sigma = \{1,2\}$ *and the top residue is given by:*

$$⨍ a|D|^{-2} := \mathrm{Res}_{z=2}\zeta_a(z) = \frac{2}{\pi}\int_{S^1}\sigma(a)\mathrm{d}\theta \qquad \forall\, a \in \mathcal{A}(S^2_{qs}) \;\; ,$$

with $\sigma : \mathcal{A}(S^2_{qs}) \hookrightarrow C(S^1)$ *the homo. defined by:* $\sigma(A) = 0$ *and* $\sigma(B) = se^{i\theta}$.

Index theory

- $\not{D}$ elliptic over M is Fredholm. If $\not{D}_V :=$ lift to a v.b. V, $\exists$ an homo.:

$$\mathrm{Index} : K^0(M) \to \mathbb{Z} \ , \qquad [V] \mapsto \mathrm{Index}(\not{D}_V) \ .$$

- To generalize it, def. of **Fredholm module** [Atiyah]: it is a triple $(\mathcal{A}, \mathcal{H}, F)$, $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, $F = F^*$, $F^2 = 1$ and $[F, \mathcal{A}] \subset \mathcal{K}(\mathcal{H})$. E.g.:

  $(\mathcal{A}, \mathcal{H}, D)$ spectral triple $\Rightarrow$ $(\mathcal{A}, \mathcal{H}, \mathrm{sign}\, D)$ is a Fredholm module.

- Let $\gamma =$ grading on $(\mathcal{A}, \mathcal{H}, D)$, $a_j \in \mathcal{A}$. The class of

$$\mathrm{ch}^F_n(a_0, \dots, a_n) = \tfrac{1}{2n!}\, \Gamma(\tfrac{n}{2} + 1)\, \mathrm{Trace}(\gamma F [F, a_0] \dots [F, a_n])$$

  in $\mathrm{PHC}^{\mathrm{ev}}(\mathcal{A})$ is indep. of n, $\forall\, n$ even and sufficiently large.

- Pairing between $\phi = (\phi_0, \phi_2, \dots) \in \mathrm{PHC}^{\mathrm{ev}}(\mathcal{A})$ and $K_0(\mathcal{A})$:

$$\langle \phi, [p] \rangle = \phi_0(p) + \textstyle\sum_{k \in \mathbb{N}} (-1)^k \, \frac{(2k)!}{k!} \, \phi_{2k}(p - \tfrac{1}{2}, p, \dots, p)$$

  $p = p^* = p^2$ is a projector. The pairing with ch^F gives:

$$K_0(\mathcal{A}) \to \mathbb{Z} \ , \quad [p] \to \langle \mathrm{ch}^F, [p] \rangle = \mathrm{Index}(pFp)$$

Charge of the q-monopole

For Podleś spheres, a Fredholm module is defined by:

$$F\,|l,m\rangle_{\pm} = |l,m\rangle_{\mp} \quad .$$

The pairing:

$$\langle \mathrm{ch}^F, [p] \rangle \equiv \mathrm{ch}^F_0(\mathrm{Trace}\, p) \quad ,$$

gives an integer valued invariant analogous to the monopole charge.

Proposition

The cohomology class of F is not trivial: $\langle \mathrm{ch}^F, [p_{qs}] \rangle = 1$.

Proved in 3 steps: 1st) prove that the charge is a continuous function of q; 2nd) a continuous function $[0,1) \to \mathbb{Z}$ is constant; 3rd) compute it for $q = 0$.

In general: evaluating $\langle \mathrm{ch}^F, [p] \rangle$ by computing kernels and cokernels (or traces) is very difficult. One needs local formulas expressing invariants through ‘integrals’.

Local index formulas

Locality makes complicated expressions computable, by neglecting irrelevant details.

A general theorem (Connes-Moscovici) applied to our case states that ch^F is cohomologous to the cocycle φ with components:

$$\varphi_0(a_0) = \mathrm{Res}_{z=0}\, z^{-1}\mathrm{Trace}(\gamma a_0|D|^{-2z}) \ \ ,$$

$$\varphi_2(a_0, a_1, a_2) \propto ⨍ \gamma a_0[D, a_1][D, a_2]|D|^{-2} \ \ .$$

In principle: φ_2 is local, φ_0 is not local.
Actually one can prove: $\varphi_2 = 0$ and $\varphi_0 = \mathrm{ch}_0^F$.

Proposition

The coboundary '$\mathrm{ch}^F - (\varphi_0, \varphi_2)$' *is zero.*

3-spheres and beyond

A different situation on the quantum 3-sphere $S^3_q := SU_q(2)$, $\mathcal{A} :=$ real representative 'functions' on $SU_q(2)$.

Two different spectral triples: a singular one [Chakraborty and Pal, K-Theory **28** (2003) 107] (scalar spinors, $q \neq 1$) and an isospectral one [Dąbrowski et al., Comm. Math. Phys. (2005), online].

If $\mathrm{ch}_1^F(a_0, a_1) := \mathrm{Trace}(a_0[F, a_1])$, then in both cases $\exists$ a local $\chi_1 : \mathcal{A}^2 \to \mathbb{C}$ such that $[\mathrm{ch}_1^F] = [\chi_1]$ in $PHC^\bullet(\mathcal{A})$.

$\Rightarrow$ $\mathrm{ch}_1^F - \chi_1 = b\psi_0 := \psi([\,.\,,\,.\,])$ for some cochain $\psi_0 : \mathcal{A} \to \mathbb{C}$.

Conjecture: Fixed $a \in \mathcal{A}$, $\psi_0(a)$ as a function of q is either 0 or a rational function $+\, q^2\partial_{q^2}$ of the Dedekind η-function.

In the 'singular' case, proved by Connes, J. Inst. Math. Jussieu **3** (2004) 17.
In the 'isospectral' case, it is yet an open problem.