Nonabelian Gerbes, Differential Geometry and Stringy Applications

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Outline

I. Anomalies in String Theory
II. String group and string structures
III. Nonabelian gerbes and their differential geometry (nonabelian B-field)
IV. Twisting nonabelian gerbes and M5-brane anomaly

III + IV

P. Aschieri, L. Cantini, B. Jurčo, CMP, 254 (2005), 367; hep-th/0312154
P. Aschieri, B. Jurčo, JHEP 0410, 068 (2004) (2005), 367; hep-th/0409200
B. Jurčo, math.DG/0510078

Heterotic String Theory

- *Perturbative space-time anomalies*

Green-Schwarz mechanism

$$dH = \mathrm{Tr}F^2 - \mathrm{Tr}R^2$$

or

$$p_1(T) - p_1(V) = 0$$

in $H^4_{\mathrm{dR}}(M) = H^4(M, \mathbb{R})$

V - $SO(32)$ or $E_8 \times E_8$ bundle (gauge theory on M)

F - curvature

$T = TM$, R - Riemannian curvature 2-form

- *Global world-sheet anomalies*

$$p_1(T) - p_1(V) = 0$$

in $H^4(M, \mathbb{Z})$

Point particle with world-line SUSY

$$I = \int dt\, \frac{1}{2} g_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{i}{2}\psi^i \left(g_{ij}\frac{d}{dt} + \frac{dx^k}{dt}\omega_{ijk}\right)\psi^j$$

$$Q = \psi_i \frac{dx^i}{dt}$$

<u>Canonical quantization</u>

$$\psi_i \sim \Gamma_i \qquad \frac{dx^i}{dt} \sim \frac{\partial}{\partial x^i}$$

$Q \sim$ Dirac operator on M

M - spin

Path integral $\sim \sqrt{D}$

$$D = i\left(\frac{d}{dt}\delta^i_j + \frac{dx^k}{dt}\omega^i_{kj}\right)$$

D - world-line Dirac operator

well defined iff M is spin

$$w_2 = 0 \in H^2(M, \mathbb{Z}_2)$$

- nonequivalent spin structures $\sim H^1(M, \mathbb{Z}_2)$

Spin structure

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin(n) \longrightarrow SO(n) \longrightarrow 1$$

• M - spin if the principal $SO(n)$-frame bundle lifts to a principal $Spin(n)$-bundle
• $\exp(iI)$ is *well* defined

• M - it not spin $w_2 \neq 0$

$$\exp(iI) \to \exp(iI)\mathrm{hol}(A)$$

A - "$U(1)$-gauge field" s.t. ambiguity in $\mathrm{hol}(A) \sim \pm 1$ cancels the ambiguity in $\sqrt{D} \pm 1$
• M - must be $\mathrm{spin}^{\mathbb{C}}$

Spin$^{\mathbb{C}}$ structure

$$1 \longrightarrow \Gamma \longrightarrow Spin(n) \times U(1) \longrightarrow Spin^{\mathbb{C}}(n) \longrightarrow 1$$

$\Gamma \subset Spin(n) \times U(1)$ generated by $(1,1)$ and $(-1,-1)$

- M - spin$^{\mathbb{C}}$ if the principal $SO(n)$-frame bundle lifts to a principal $Spin^{\mathbb{C}}(n)$-bundle
- $\exp(iI)\mathrm{hol}(A)$ is *well* defined
- there exist a line bundle L such that

$$w_2(M) = c_1(L) \mathrm{mod} 2$$

in $H^2(M, \mathbb{Z}_2)$ or

-

$$W_3(M) = 0$$

in $H^3(M, \mathbb{Z})$

D-branes

We can have a D-brane which is *not* spin$^{\mathbb{C}}$,
i.e $W_3 \neq 0$

- but

$$a(D) = W_3(D) - [H]|_D = 0$$

in $H^3(D, \mathbb{Z})$ *Freed - Witten* anomaly

- recall

$$H^3(M, \mathbb{Z}) \sim \text{[abelian 1-gerbes on M]}$$

D-brane - *trivialization* of an abelian 1-gerbe defined by $a(D)$

• if $a(D) \neq 0$ but

$$na(D) = 0 \qquad \text{(torsion)}$$

this corresponds to an stack on n D-branes
we need at least $U(n)$-“gauge field (bundle)” to cancel the anomaly
stack of D-banes $\sim$ module of the abelian 1-gerbe
• in general $U(\mathcal{H})$

M-theory, M5-branes

M5-brane with M2-branes ending on it

$$a(M5) = \text{``}W_4(M5) - [G_4]|_{M5}\text{''} = 0$$

in $H^4(M5, \mathbb{Z})$ *Diaconescu - Freed - Witten* anomaly

- recall

$$H^4(M, \mathbb{Z}) \sim \text{[abelian 2-gerbes on M]}$$

$M5$-brane - *trivialization* of an abelian 2-gerbe defined by $a(M5)$

- coinciding $M5$-branes

$$a(D) \neq 0$$

and we need a nonabelian gerbe to cancel the anomaly

- stack of M5 branes $\sim$ module of the abelian 2-gerbe

String stuctures - return to heterotic anomaly

$$p_1(T) - p_1(V)$$

in $H^4(M, \mathbb{Z})$

- Witten, Killingback

Dirac-Ramond operator $\sim$ Dirac operator on the loop space LM

LM must be $spin$

- frame bundle (principal $SO(n)$-bundle) $P \to M \Rightarrow$ principal $LSO(n)$-bundle $LP \to LM$

$$1 \longrightarrow S^1 \longrightarrow \widetilde{LSO(n)} \longrightarrow LSO(n) \longrightarrow 1$$

obstruction to lift LP into a principal $\widetilde{LSO(n)}$-bundle is the *transgression* of $p_1(T)$ in $H^3(LM, \mathbb{Z})$

- M (TM) has a string structure if this obstruction vanishes (more generally $T - V$ makes sense in K-theory, it has a string structure if $p_1(T) - p_1(V)$ vanishes in $H^4(M, \mathbb{Z})$)

String Group

π_k	0	1	2	3	4	5	6	7
$O(n)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$SO(n)$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$Spin(n)$	0	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$String(n)$	0	0	0	0	0	0	0	$\mathbb{Z}$

• *not* Lie; defined up to homotopy

$$1 \longrightarrow K(\mathbb{Z},2) \longrightarrow String(n) \longrightarrow Spin(n) \longrightarrow 1$$

such that the boundary map

$$\pi_3 Spin(n) \to \pi_2 K(\mathbb{Z},2) = \mathbb{Z}$$

is given by

$$p_1/2 \in H^4(BSpin(n)) \sim \mathrm{Hom}(\pi_3 Spin(n), \mathbb{Z})$$

• more generally $Spin(n) \to G$-compact simply connected $String(n) \to String_G$

• String structure $\sim$ lifting of a principal G-bundle P to a principal $String_G$-bundle ($\frac{1}{2}p_1(P) = 0$)

• models of $String$ - Brylinski; Teichner, Stolz; Baez, Crans, Schreiber, Stevenson; Henriquez

Crossed Module $(H \longrightarrow D)$

H,D Lie groups. H is a crossed D-module if there is a group homomorphism $\alpha : H \longrightarrow D$ and an action of D on H, $(d, h) \mapsto {}^{d}h$, such that

$$^{\alpha(h)}h' = hh'h^{-1} \;\text{ for }\; h, h' \in H$$

and

$$\alpha({}^{d}h) = d\alpha(h)d^{-1} \;\text{ for }\; h \in H, d \in D.$$

holds true.

- BCSS constuction of $String$

$$H = \widetilde{\Omega G} \quad \text{and} \quad D = P_0 G$$

Nonabelian (bundle) gerbes

$\{O_\alpha\}$ open covering of M

- Local description - *Nonabelian Čech 2-cocycle*

$$\{d_{\alpha\beta}, h_{\alpha\beta\gamma}\}$$

$$d_{\alpha\beta} : O_\alpha \cap O_\beta \to D$$

$$h_{\alpha\beta\gamma} : O_\alpha \cap O_\beta \cap O_\gamma \to H$$

fulfilling the following cocycle condition

$$d_{\alpha\beta} d_{\beta\gamma} = \alpha(h_{\alpha\beta\gamma}) d_{\alpha\gamma} \text{ on } O_\alpha \cap O_\beta \cap O_\gamma$$

and

$$h_{\alpha\beta\gamma} h_{\alpha\gamma\delta} = {}^{d_{\alpha\beta}} h_{\beta\gamma\delta} h_{\alpha\beta\delta} \text{ on } O_\alpha \cap O_\beta \cap O_\gamma \cap O_\delta.$$

• Stable equivalence

$\{d_{\alpha\beta}, h_{\alpha\beta\gamma}\}$ and $\{d'_{\alpha\beta}, h'_{\alpha\beta\gamma}\}$ are related as

$$d'_{\alpha\beta} = d_\alpha \alpha(h_{\alpha\beta}) d_{\alpha\beta} d_\beta^{-1}$$

and

$$h'_{\alpha\beta\gamma} = {}^{d_\alpha}h_{\alpha\beta}\, {}^{d_\alpha d_{\alpha\beta}}h_{\beta\gamma}\, {}^{d_\alpha}h_{\alpha\beta\gamma}\, {}^{d_\alpha}h_{\alpha\gamma}^{-1}$$

with $d_\alpha : O_\alpha \to D$ and $h_{\alpha\beta} : O_\alpha \cap O_\beta \to H$.

- [Principal $String_G$-bundles] $\sim$ [$(\widetilde{\Omega G} \to P_0G)$-gerbes]

- [Principal G-bundles] $\sim$ [$(\Omega G \to P_0G)$-gerbes]

- *String structure*

- lift of a $(\Omega G \to P_0G)$-gerbe to a $(\widetilde{\Omega G} \to P_0G)$-gerbe

Gerbe connection

$\{A_\alpha, a_{\alpha\beta}\}$

$A_\alpha \in \Omega^1(O_\alpha) \otimes \mathrm{Lie}(D)$

$a_{\alpha\beta} \in \Omega^1(O_\alpha \cap O_\beta) \otimes \mathrm{Lie}(H)$ fulfilling

$$A_\alpha = d_{\alpha\beta} A_\beta d_{\alpha\beta}^{-1} + d_{\alpha\beta} d d_{\alpha\beta}^{-1} + \alpha(a_{\alpha\beta})$$

on $O_\alpha \cap O_\beta$ and

$$a_{\alpha\beta} + {}^{d_{\alpha\beta}} a_{\beta\gamma} = h_{\alpha\beta\gamma} a_{\alpha\gamma} h_{\alpha\beta\gamma}^{-1} + h_{\alpha\beta\gamma} d h_{\alpha\beta\gamma}^{-1} + T_{A_\alpha}(h_{\alpha\beta\gamma}^{-1})$$

on $O_\alpha \cap O_\beta \cap O_\gamma$.

$T_A(h)$ - $\mathrm{Lie}(H)$-valued one-form. For $X \in \mathrm{Lie}(D)$ we put $T_X(h) = [h^{\exp(tX)}(h^{-1})]$, [ ] - tangent vector to the curve at the group identity 1_H. For $\mathrm{Lie}(D)$-valued one form $A = A^\rho X^\rho$, with $\{X^\rho\}$ a basis of $\mathrm{Lie}(D)$, put $T_A \equiv A^\rho T_{X^\rho}$.

- curvature $F_\alpha \in \Omega^2(O_\alpha) \otimes \mathrm{Lie}(D)$

$$F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha$$

Nonabelian B-field

$\{B_\alpha, \delta_{\alpha\beta}\}$
$B_\alpha \in \Omega^2(O_\alpha) \otimes \mathrm{Lie}(H)$
$\delta_{\alpha\beta} \in \Omega^2(O_{\alpha\beta}) \otimes \mathrm{Lie}(H)$ such that

$$B_\alpha = {}^{d_{\alpha\beta}}B_\beta + \delta_{\alpha\beta} \text{ on } O_\alpha \cap O_\beta$$

and

$$\delta_{\alpha\beta} + {}^{d_{\alpha\beta}}\delta_{\beta\gamma} = h_{\alpha\beta\gamma}\delta_{\alpha\gamma}h^{-1}_{\alpha\beta\gamma} + B_\alpha - h_{\alpha\beta\gamma}B_\alpha h^{-1}_{\alpha\beta\gamma}$$

on $O_\alpha \cap O_\beta \cap O_\gamma$.

- $$H_\alpha = dB_\alpha + T_{A_\alpha}(B_\alpha)$$

Twisting by an abelian 2-gerbe

$H = \widetilde{\Omega G}$, $D = P_0 G$

$$h_{\alpha\beta\gamma} h_{\alpha\gamma\delta} \lambda_{\alpha\beta\gamma\delta} = {}^{d_{\alpha\beta}} h_{\beta\gamma\delta} h_{\alpha\beta\delta}$$

$$a_{\alpha\beta} + {}^{d_{\alpha\beta}} a_{\beta\gamma} - \alpha_{\alpha\beta\gamma}$$

$$= h_{\alpha\beta\gamma} a_{\alpha\gamma} h^{-1}_{\alpha\beta\gamma} + h_{\alpha\beta\gamma} d h^{-1}_{\alpha\beta\gamma} + T_{A_\alpha}(h^{-1}_{\alpha\beta\gamma})$$

$$B_\alpha + \beta_{\alpha\beta} = {}^{d_{\alpha\beta}} B_\beta + \delta_{\alpha\beta}$$

$$H_\alpha - \gamma_\alpha = dB_\alpha + T_{A_\alpha}(B_\alpha)$$

- $\lambda_{\alpha\beta\gamma\delta}$ (U(1)-valued) is the obstruction to the $String$ structure - abelian 3-cocycle - abelian 2-gerbe
- $\{\lambda_{\alpha\beta\gamma\delta}, \alpha_{\alpha\beta\gamma}, \beta_{\alpha\beta}, \gamma_\alpha\}$ is a *Deligne* class - abelian 2-gebre with curvings (α, β, γ are u(1)-valued).

- Diaconescu-Freed-Witten anomaly of a stack of M5-branes can be canceled by a twisted nonabelian gerbe with $G = E_8$
- field content of a higher gauge theory
- gauge theory with a nonabelian 2-form gauge potential B on the M5 world-volume? Action?
- nonabelian B couples to the ends of M2-branes ending in M5-branes similarly as in one degree lower non-abelian gauge potential A couples to the ensds of srings ending on the D-branes.