GAUGE/GRAVITY DUALS AND GENERALISED COMPLEX GEOMETRY

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INTRODUCTION

• Supersymmetric backgrounds with non-zero fluxes play a crucial role in trying to connect string theory to $4d$ physics. For instance

- string compactifications
  $\longrightarrow$ potential to fix some of the moduli
  $\longrightarrow$ further breaking of supersymmetry
- generate vacua with positive cosmological constant
- generalisations of the AdS/CFT correspondence
  $\longrightarrow$ supergravity solutions describing branes wrapping non-trivial cycles
- landscape of vacua
- spontaneous supersymmetry breaking in brane-world models

STRUCTURE OF THE TALK

- The fluxes backreact on the geometry $\rightarrow$ new compactification manifolds

Natural questions:

- characterisation of the allowed geometries
  - G-structures
  - Generalised Complex Geometry
- applications to AdS/CFT: construction of explicit examples of supersymmetric backgrounds with fluxes
  - examples of old and new solutions dual to $\mathcal{N} = 1$ gauge theories in 4 dimensions

GENERAL FORM OF SUGRA SOLUTIONS

- Look for solutions of Type II supergravity in 10d of the form
  - the space time has product structure

$$M_{10} = X_4 \times_w Y_6$$

4d Minkowski 6d internal manifold (compact)

$\longrightarrow$ the metric is

$$\mathrm{d}s_{10}^2 = e^{2A(y)}\mathrm{d}s_4^2 + \mathrm{d}s_6^2$$

$\longrightarrow$ the aim is to obtain a 4d theory with Poincaré invariance

  - non-zero background values of some RR and/or NS fields

$$F_P \neq 0$$

$\longrightarrow$ necessary to have realistic models

- The geometry of the internal manifold determines

$\longrightarrow$ field content and symmetries of the 4d low energy theory

SUPERSYMMETRIC SOLUTIONS

- For SUSY solutions corresponding to product spaces

10d equations of motion $\Leftrightarrow$

- vanishing of the spinor variations

  $\delta\psi_M = 0$ gravitino

  $\delta\lambda = 0$ dilatino

- Bianchi identities and e.o.m. for the forms

  $\mathrm{d}F_P = 0$

  $\mathrm{d}(*F_P) = 0$

- SUSY variations $\longrightarrow$ necessary conditions to have SUSY vacua
- Bianchi and e.o.m $\longrightarrow$ further constraints on the solutions (no-go theorem)

PURE GEOMETRY: CALABI YAU COMPACTIFICATIONS

- SUSY variations

$$\delta\psi_M^{(i)} = \nabla\epsilon^{(i)} = 0 \qquad\qquad (\nabla_M = \partial_M + \frac{1}{2}\omega_M^{AB}\Gamma_{AB})$$

$$\delta\lambda^{(i)} = 0$$

- $4d$ times $6d$ splitting

$$\delta\psi_M = \nabla_M\epsilon = 0 \begin{cases} \partial_\mu\zeta_+ = 0 & \text{(constant spinor)} \\ \nabla_m\eta = 0 & \text{(covariantly constant spinor)} \end{cases}$$

- $\nabla_m\eta = 0$ implies
  - number of $4d$ supersymmetries: one solution $\Leftrightarrow$ one SUSY parameter in $4d$
  - information about the geometry of the internal manifold: Calabi-Yau manifold

    $\rightarrow$ $SU(3)$ holonomy

    $\rightarrow$ Ricci flat $\quad 0 = [\nabla_m, \nabla_n]\eta_+ = \frac{1}{4}R_{mn}{}^{pq}\gamma_{pq}\eta_+ \Rightarrow R_{mn} = 0$

G-STRUCTURE vs HOLONOMY

- Back to the Calabi Yau condition

$$\nabla_m \eta_+ = 0$$

$$\Downarrow$$

2 types of constraints

$$\swarrow \qquad \searrow$$

topological

$\exists$ a globally defined non-vanishing invariant spinor

$$\eta_+$$

$$\Updownarrow$$

reduction of the structure group (G-Structure)

$$SO(6) \to SU(3)$$

$$\Updownarrow$$

$\exists$ globally defined invariant tensors

$$J = -2i\eta_+^\dagger \gamma_{mn} \eta_+$$

$$\Omega = -2i\eta_-^\dagger \gamma_{mnp} \eta_+$$

differential

the spinor is covariantly constant

$$\nabla_m \eta_+ = 0$$

$$\Updownarrow$$

reduced holonomy (special holonomy)

$$SO(6) \to SU(3)$$

$$\Updownarrow$$

the forms are closed

$$\mathrm{d}J = 0$$

$$\mathrm{d}\Omega = 0$$

$\Rightarrow$ a G-structure is the topological remnant of special holonomy

FLUX VACUA

- Turn on non-zero fluxes
  - $F_p \neq 0$ modify the eqs. of motion

$$R_{MN} \sim \sum_p F_{MQ_1 \ldots Q_p} F_N^{\ Q_1 \ldots Q_p}$$

  - SUSY variations ($6d$ part)

$$\delta\psi_m = D_m\eta = \nabla_m\eta + G_m\eta = 0$$

Levi Civita fluxes

  - the geometry changes
    $\longrightarrow$ warped product

    $\longrightarrow$ the internal manifold is no longer Calabi-Yau

$$[D_m, D_n]\eta = \frac{1}{4}\hat{R}_{mn}{}^{pq}\gamma_{pq}\eta \neq 0 \qquad (\hat{R}_{[mnp]q} \neq 0)$$

- Can we still say something about the geometry of the internal manifold ?

G-STRUCTURES AND FLUX COMPACTIFICATIONS

- In the presence of fluxes
  - topological conditions are still valid
    - $\rightarrow$ $\exists$ a globally defined spinor
    - $\rightarrow$ $\exists$ a G-structure
    - $\rightarrow$ implicit in the existence of a $4d$ effective action (4 $\times$ 6 splitting)
  - differential conditions
    - $\rightarrow$ set of differential constraints on the spinor/forms
- Reinterpret the SUSY variations

$$D_m \eta_+ = (\nabla_m + \not{F}_m)\eta_+ = 0 \qquad \text{SUSY}$$

$$D^T_m \eta_+ = (\nabla_m + T_m)\eta_+ = 0 \qquad \text{G-structure}$$

$D^T_m \rightarrow$ connection with torsion $(D^T_m \eta_+ = (\nabla_m - \frac{3!}{4} T_{[mnp]}\gamma^{np})\eta_+)$

$\Rightarrow$ for SUSY vacua the torsion must be cancelled by fluxes

$$D_m \eta_+ = 0 \Rightarrow (\not{F}_m + T_m)\eta_+ = 0$$

$\Rightarrow$ classify the internal manifolds depending on the non-zero torsion components

THE SUPERSYMMETRY VARIATIONS

- SUSY transformations for IIA/IIB

$$\delta\psi_M = (D_M\epsilon + \frac{1}{4}H_M\mathcal{P})\epsilon + \frac{1}{16}e^{\phi}\sum_n \not{F}^{(2n)}\,\Gamma_M\mathcal{P}_n\,\epsilon$$

$$\delta\lambda = (\not\partial\phi + \frac{1}{2}\not{H}\mathcal{P})\epsilon + \frac{1}{8}e^{\phi}\sum_n(-1)^{2n}(5-2n)\not{F}^{(2n)}\,\mathcal{P}_n\epsilon$$

$\epsilon = (\epsilon^1, \epsilon^2) \to$ Majorana-Weyl spinors

IIA $\to \mathcal{P} = \Gamma_{11}$, $\mathcal{P}_n = \Gamma_{11}\sigma_1$ IIB $\to \mathcal{P} = -\sigma_3$, $\mathcal{P}_n = \sigma_1$ ($n+1/2$ even), $i\sigma_2$ ($n+1/2$ odd))

- Spinor ansatz
  - product space $\to SO(1,9) \to SO(1,3)\times SO(6)$
  - decomposition of the 10d spinors

IIA $\to$

$$\epsilon_1 = \zeta_+\otimes\eta^1_+ + \zeta_-\otimes\eta^1_-$$
$$\epsilon_2 = \zeta_+\otimes\eta^2_- + \zeta_-\otimes\eta^2_+$$

IIB $\to$

$$\epsilon_1 = \zeta_+\otimes\eta^1_+ + \zeta_-\otimes\eta^1_-$$
$$\epsilon_2 = \zeta_+\otimes\eta^2_+ + \zeta_-\otimes\eta^2_-$$

with $\zeta_\pm \to 4d$ chiral spinor ($\zeta_+^* = \zeta_-$)

$\eta^{(i)}_\pm = \sum_{a=1}^4 c_a^{(i)}\theta_a \to 6d$ chiral spinor ($\eta^{i*}_+ = \eta^i_-$)

G-STRUCTURES AND $\mathcal{N}=1$ COMPACTIFICATIONS

- There are different G-structures compatible with $\mathcal{N}=1$ SUSY in 4d $\longrightarrow$ depends on the relation between the internal spinors η^1_+ and η^2_-
  - SU(3) structure $\longrightarrow$ η^1_+ and η^2_- are parallel

$$\begin{cases} \eta^1_+ = a\eta_+ \\ \eta^2_+ = b\eta_+ \end{cases} \qquad a, b \quad \text{complex function on Y}$$

- definitions
  - 1 invariant spinor $\longrightarrow$ η_+
  - invariant real 2-form and holomorphic (3,0)-form

$$J = -i\eta^\dagger_+\gamma_{mn}\eta_+ \mathrm{d}x^m \wedge \mathrm{d}x^n \qquad J\wedge\Omega = 0$$

$$\Omega = -\tfrac{i}{3}\eta^\dagger_-\gamma_{mnp}\eta_+ \mathrm{d}x^m \wedge \mathrm{d}x^n \wedge \mathrm{d}x^p \qquad \frac{1}{3!}J^3 = \frac{i}{8}\Omega\wedge\bar{\Omega}$$

- torsion classes $\longrightarrow$ deviation from the Calabi Yau condition

$$\begin{aligned} \mathrm{d}J &= \frac{3}{2} Im(\bar{W}_1\Omega) + W_4\wedge J + W_3 \\ \mathrm{d}\Omega &= W_1\, J\wedge J + W_2\wedge J + \bar{W}_5\wedge\Omega \end{aligned}$$

W_i $\longrightarrow$ torsion in SU(3) components

- manifolds are classified in terms of the non-zero components of the intrinsic torsion

Ex:	complex	$\mathcal{W}_1 = \mathcal{W}_2 = 0$	$\Rightarrow$	$\mathrm{d}J = W_4 \wedge J + W_3$
				$\mathrm{d}\Omega = \bar{W}_5 \wedge \Omega$
	symplectic	$\mathcal{W}_1 = \mathcal{W}_3 = 0$	$\Rightarrow$	$\mathrm{d}J = W_4 \wedge J$
				$\mathrm{d}\Omega = W_2 \wedge J + \bar{W}_5 \wedge \Omega$
	Kähler	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0$	$\Rightarrow$	$\mathrm{d}J = 0$
				$\mathrm{d}\Omega = \bar{W}_5 \wedge \Omega$
	Calabi-Yau	$\mathcal{W}_i = 0 \, i = 1, \ldots, 5$	$\Rightarrow$	$\mathrm{d}J = \mathrm{d}\Omega = 0$

- general conditions for $\mathcal{N} = 1$ vacua with SU(3) structure [grana, minasian, m.p., tomasiello]

IIB	complex	$\mathrm{d}\Omega = \mathrm{W}_5 \wedge \Omega$
IIA	twisted symplectic	$dJ = W_4 \wedge J + H^{(6)}$

- SU(2) structure $\longrightarrow \eta^1_+$ and η^2_- are never parallel

$$\begin{cases} \eta^1_+ = a\eta_+ \\ \eta^2_+ = bz \cdot \eta_- \end{cases} \qquad a, b \quad \text{complex function on Y}$$

  - definitions
    - 2 invariant spinors $\longrightarrow \eta_+$ and $z \cdot \eta_-$
    - 1 complex vector, 1 invariant real 2-form and 1 holomorphic (2,0)-form

$$z = \eta^\dagger_- \gamma_m \eta_+ \mathrm{d}x^m \qquad \omega \wedge j = 0$$

$$j = -i\eta^\dagger_+ \gamma_{mn} \eta_+ \mathrm{d}x^m \wedge \mathrm{d}x^n \qquad zj = z\omega = 0$$

$$\omega = -\tfrac{i}{3}\eta^\dagger_- \gamma_{mnp} \eta_+ \mathrm{d}x^m \wedge \mathrm{d}x^n$$

  - almost product structure (4 $\times$ 2 splitting)
  - torsion classes and SUSY conditions $\rightarrow$ too many SU(2) representations ... [dall'agata]

- degenerate SU(2) structure $\rightarrow \eta_+^1$ and η_-^2 can be parallel at some points on the manifold

$$\begin{cases} \eta_+^1 = a\eta_+ \\ \eta_+^2 = c_1\eta_+ + c_2 z\cdot\eta_- \end{cases} \qquad c_i \quad \text{complex function on Y}$$

- Is there a way to treat all these cases on the same ground?

  how to obtain general conditions for $\mathcal{N}=1$ vacua?

  $\rightarrow$ Generalised Complex Geometry

GENERALISED COMPLEX GEOMETRY

[hitchin, gualtieri, ...]

- Formalism that treats the tangent and cotangent bundles on the same footing

$$\mathcal{X} = (v + \xi) \in T(Y) \oplus T^*(Y)$$

$\rightarrow$ extend differential geometry to $T \oplus T^*$

- metric on $T \oplus T^*$
  - natural metric $\rightarrow$ pairing of vectors and forms

$$< v_1 + \xi_1, v_2 + \xi_2 >= \frac{1}{2}(\xi_1(v_2) + \xi_2(v_1))$$

in matrix form

$$\mathcal{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  - does not need the metric on Y
  - metric plus orientation reduce the structure group to O(6,6)

GENERALISED ALMOST COMPLEX STRUCTURE

- Generalised almost complex structure
  - linear map from $T \oplus T^*$ to itself

$$\mathcal{J} : T \oplus T^* \longrightarrow T \oplus T^*$$

$$\mathcal{J}^2 = -1_{12}$$

$$\mathcal{J}^\dagger \mathcal{I} \mathcal{J} = \mathcal{I} \qquad \text{hermiticiy}$$

  - the structure group is reduced to U(3,3)
  - unified description of complex and symplectic geometry

$$\mathcal{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^\dagger \end{pmatrix} \qquad J_n^m \text{ almost complex structure}$$

$$\mathcal{J}_2 = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix} \qquad J_n^m \text{ pre-symplectic structure}$$

INTEGRABILITY CONDITION

• ∃ an analogue of the integrability condition (Courant bracket) → generalised complex manifolds

	T	$T \oplus T^*$
almost complex structure	J $J^2 = -1_6$	$\mathcal{J}$ $\mathcal{J}^2 = -1_{12}$
projectors	$\pi_\pm = (1_6 \pm iJ)/2$	$\Pi_\pm = (1_{12} \pm i\mathcal{J})/2$
integrability	$\pi_+[\pi_-(v_1), \pi_-(v_2)]_{\text{Lie}} = 0$	$\Pi_+[\Pi_-(\mathcal{X}_1), \Pi_-(\mathcal{X}_2)]_{\text{C}} = 0$

• On $T \oplus T^*$ there is no Lie bracket → Courant bracket which satisfies Jacobi on the $\pm i$-eigenvalues of $\mathcal{J}$

$$[v_1 + \xi_1, v_2 + \xi_2]_{\text{C}} = [v_1, v_2]_{\text{Lie}} + \{L_{v_1}\xi_2 - \frac{1}{2}d(\iota_{v_1}\xi_2) - (1 \leftrightarrow 2)\}$$

PURE SPINORS

- Clifford algebra on $T \oplus T^*$ → Cliff(6,6)

$$\{\gamma^m, \gamma^n\} = 0 \,, \qquad \{\gamma^m, \gamma_n\} = \delta^m_n \,, \qquad \{\gamma_m, \gamma_n\} = 0 \,.$$

- Cliff(6,6) spinors
  - sum of (p,q) forms → $\varphi = \sum_{p,q} \omega^{(p,q)} \gamma^{m_1 \ldots m_p \bar{n}_1 \ldots \bar{n}_q} \eta_+ \in \Lambda^\bullet T^*$
  - gamma matrix representation → $\gamma^m = \mathrm{d}x^m, \qquad \gamma_m = \iota_m$

- Pure spinors
  - vacuum of the Clifford algebra (annihilated by half of the gammas)
  - 1 pure spinor reduces the structure group to SU(3,3)
  - compatible pure spinors → have three common annihilators
    → reduce the structure to SU(3) $\times$ SU(3)

- Ex: on a manifold of $SU(3)$ structure ∃ 2 compatible natural pure spinors

$$\Phi_+ = e^{-iJ} = 1 - iJ - \frac{1}{2}J^2 + \frac{i}{6}J^3 \qquad \text{annihilated by } \gamma_m + iJ_{mn}\gamma^n$$

$$\Phi_- = \Omega \qquad \text{annihilated by } \gamma^i, \gamma_{\bar{i}}$$

PURE SPINORS AND GENERALISED COMPLEX STRUCTURES

- 1 to 1 correspondence between generalised complex structures and pure spinors
  - topological level

    annihilator of Φ ($\mathcal{X}_i \Phi = 0$) = i - eigenspace of $\mathcal{J}$

  - Ex: SU(3) structure manifold

$$\begin{array}{ccc} \mathcal{J}_1 & \longleftrightarrow & \Omega \\ \mathcal{J}_2 & \longleftrightarrow & e^{iJ} \end{array}$$

  - differential level

    $\mathcal{J}$ integrable = $\exists$ a vector and a 1-form s.t.

$$\mathrm{d}\Phi = (v + \xi \wedge)\Phi$$

- Generalised Calabi Yau condition

$$\mathrm{d}\Phi = 0$$

  - twist with a three-form $\rightarrow$ twisted Generalised Calabi Yau

$$\mathrm{d}\Phi + H \wedge \Phi = 0$$

PURE SPINORS AND $\mathcal{N}=1$ COMPACTIFICATIONS

[grana, minasian, m.p., tomasiello]

- Pure spinors can be seen as bispinors $\longrightarrow$ Clifford map

$$\sum \frac{1}{k!} C^{(k)}_{i_1 \ldots i_k} dx^{i_i} \wedge \ldots \wedge dx^{i_k} \leftrightarrow \sum \frac{1}{k!} C^{(k)}_{i_1 \ldots i_k} \gamma^{i_i \ldots i_k}$$

- define Cliff(6,6) spinors by fierzing the SUSY parameters η^1 and η^2

$$\Phi_+ = \eta^1_+ \otimes \eta^{2\,\dagger}_+ \qquad \text{sum of even forms}$$
$$\Phi_- = \eta^1_+ \otimes \eta^{2\,\dagger}_- \qquad \text{sum of odd forms}$$

- treat Cliff(6,6) spinors and RR fluxes on the same ground

- use Cliff(6) gamma matrices to prove that Φ_+ and Φ_- are pure

$$(\delta + iJ_1)_m^{\ n} \gamma_n \eta^1_+ \otimes \eta^{2\,\dagger}_\pm = 0 \qquad (J_{1\,mn} = -2i\eta^{1\,\dagger}_+ \gamma_{mn} \eta^1_+)$$
$$\eta^1_+ \otimes \eta^{2\,\dagger}_\pm \gamma_n (\delta \mp iJ_2)_m^{\ n} = 0 \qquad (J_{2\,mn} = -2i\eta^{2\,\dagger}_+ \gamma_{mn} \eta^2_+)$$

- $\mathcal{N} = 1$ vacua have two natural pure spinors

$$\Phi_+ = \eta^1_+ \otimes \eta^{2\,\dagger}_+ = \frac{a\bar{b}}{8}(\bar{c}_1 e^{-ij} - i\bar{c}_2 \omega) \wedge e^{z\wedge\bar{z}/2}$$

$$\Phi_- = \eta^1_+ \otimes \eta^{2\,\dagger}_- = -i\frac{ab}{8}(\bar{c}_2 e^{-ij} + i\bar{c}_1 \omega) \wedge z$$

  - SU(3) $\times$ SU(3) structure on $T \oplus T^*$
  - does not need a definite structure on T

- Special cases
  - SU(3) structure on T

$$\Phi_+ = \frac{a\bar{b}}{8} e^{-iJ} \qquad \Phi_- = -i\frac{ab}{8}\Omega$$

  - static SU(2) structure on T

$$\Phi_+ = \frac{a\bar{b}}{8}\omega \wedge e^{z\wedge\bar{z}/2} \qquad \Phi_- = -i\frac{ab}{8} e^{-ij} \wedge z$$

EQUATIONS FOR THE PURE SPINORS

- Derive a set of differential conditions on the pure spinors from the SUSY variations
  - $\Phi_\pm$ are bispinors $\rightarrow$ express their exterior derivative in terms of the covariant derivatives of η^1 and η^2

$$\mathrm{d}\Phi_\pm = \mathrm{d}x^m \nabla_m \Phi_\pm = \mathrm{d}x^m \Big((\nabla_m \eta^1_+) \otimes \eta^{2\,\dagger}_\pm + \eta^1_+ \otimes (\nabla_m \eta^{2\,\dagger}_\pm \Big)$$

  - use the SUSY variations for η^1 and η^2 $\rightarrow$ two differential equations

IIA $\rightarrow$

$$e^{-2A+\phi}(d - H\wedge)(e^{2A-\phi}\Phi_+) = 0$$

$$e^{-2A+\phi}(d - H\wedge)(e^{2A-\phi}\Phi_-) = dA \wedge \bar{\Phi}_- + \frac{1}{16} e^{\phi}\Big[(|a|^2 - |b|^2)F_{\mathrm{IIA}} - i(|a|^2 + |b|^2) * F_{\mathrm{IIA}}\Big]$$

$$F_{\mathrm{IIA}} = F_0 - F_2 + F_4 - F_6$$

IIB $\rightarrow$

$$e^{-2A+\phi}(d - H\wedge)(e^{2A-\phi}\Phi_-) = 0$$

$$e^{-2A+\phi}(d - H\wedge)(e^{2A-\phi}\Phi_+) = dA \wedge \bar{\Phi}_+ + \frac{1}{16} e^{\phi}\Big[(|a|^2 - |b|^2)F_{\mathrm{IIB}} - i(|a|^2 + |b|^2) * F_{\mathrm{IIB}}\Big]$$

$$F_{\mathrm{IIB}} = F_1 + F_3 + F_5$$

INTERPRETATION

- $\mathcal{N} = 1$ vacua are characterised by the differential properties of two compatible pure spinors on $T \oplus T^*$
  - one spinor is twisted closed

$$\left.\begin{array}{ll} \text{IIA} & (d - H\wedge)\Phi_+ = 0 \\ \text{IIB} & (d - H\wedge)\Phi_- = 0 \end{array}\right\} \Longrightarrow \text{twisted generalised Calabi Yau}$$

  - the RR fields act as torsion

$$\begin{array}{ll} \text{IIA} & (d - H\wedge)\Phi_- = F_{RR} \\ \text{IIB} & (d - H\wedge)\Phi_+ = F_{RR} \end{array}$$

	T	$T \oplus T^*$
spinors	$(0, q)$ forms	(p, q) forms
pure spinor	η_0 vacuum of Cliff(6)	Φ vacuum of Cliff(6,6)
	$\nabla_m \eta_0 = 0$	$\mathrm{d}\Phi = 0$
	Calabi Yau	Generalised Calabi Yau

GAUGE/GRAVITY DUALS AND GCY

- Gravity duals of supersymmetric gauge theories
  - solutions with fluxes on the internal manifold
  - the 6d internal manifold is non-compact (evade the no-go theorem)

$\longrightarrow$ examples of Generalised Calabi Yau

- Method to find new solutions
  - choose a G-structure $\rightarrow$ only information about the spinor
  - trade SUSY variation for a set of algebraic and differential equations for representations of G (pure spinor equations)
  - make a suitable ansatz for the metric and the fluxes

IIB SOLUTIONS WITH SU(3) STRUCTURE

- Choice of the SU(3) structure
  - spinors $\longrightarrow$ η^1_+ and η^2_- are parallel

$$\begin{cases} \eta^1_+ = a\eta_+ \\ \eta^2_+ = b\eta_+ \end{cases}$$

  - pure spinors

$$\Phi_+ = \frac{a\bar{b}}{8} e^{-iJ} \qquad \Phi_- = -i\frac{ab}{8}\Omega$$

- Solutions
  - expand the pure spinor equations $\rightarrow$ general necessary for $\mathcal{N} = 1$ solutions
  - solve Bianchi end e.o.m for the fluxes

- Expanding the pure spinors equations in SU(3) representations → set of general necessary for $\mathcal{N}=1$ solutions
  - equation for Ω

$$\left.\begin{aligned} &W_1 = H^{(1)} = 0 \\ &W_2 = 0 \\ &\mathrm{d}(2A - \varphi + \log ab) = -\bar{W}_5 \end{aligned}\right\} \Longrightarrow$$

complex manifolds
$\mathcal{W}_1 = \mathcal{W}_2 = 0$

  - equation for e^{-iJ}

$$a\bar{b}(W_3 - iH^{(6)}) = \frac{i}{2}e^{\varphi}[(|a|^2 - |b|^2)F_3^{(6)} - i(|a|^2 + |b|^2) * F_3^{(6)}]$$

$$a\bar{b}\mathrm{d}(2A - \varphi + \log a\bar{b}) - \bar{a}b\mathrm{d}A = \frac{1}{2}e^{\varphi}[(|a|^2 - |b|^2)F_1^{(3)} - i(|a|^2 + |b|^2)(*F_5)^{(3)}]$$

$$a\bar{b}\mathrm{d}(2A - \varphi + \log \bar{a}b) - \bar{a}b\mathrm{d}A + 2a\bar{b}(W_4 - iH^{(3)}) = -e^{\varphi}[(|a|^2 - |b|^2)F_5^{(3)} - i(|a|^2 + |b|^2)(*F_1)^{(3)}]$$

$$a\bar{b}\mathrm{d}(2A - \varphi + \log \bar{a}b) + \bar{a}b\mathrm{d}A + a\bar{b}(W_4 - iH^{(3)}) = \frac{i}{2}e^{\varphi}[(|a|^2 - |b|^2)F_3^{(3)} + i(|a|^2 + |b|^2)(*F_3)^{(3)}]$$

$$F_3^{(1)} = 0$$

KNOWN SOLUTIONS

- The two regular solutions dual to $\mathcal{N}=1$ Super Yang-Mills are GCY with SU(3) structure
  - Klebanov-Strassler solution (fractional D3-branes on the deformed conifold): type B
  - Maldacena-Nunez solution (wrapped D5-brane) : type C :

	$a = \pm ib$ (B)	$a = \pm b$ (C)
1	$W_1 = F_3^{(1)} = H^{(1)} = 0$	
8	$W_2 = 0$	
6	$W_3 = 0$ $e^\phi F_3^{(6)} = *H^{(6)}$	$H^{(6)} = 0$ $W_3 = \pm e^\phi * F_3^{(6)}$
3	$e^\phi F_5^{(\bar{3})} = \frac{2}{3} i \bar{W}_5 = i W_4 =$ $-2i\bar{\partial} A = -4i\bar{\partial}\log\alpha, \quad \bar{\partial}\phi = 0$	$\pm e^\phi F_3^{(\bar{3})} = 2i\bar{W}_5 =$ $-2i\bar{\partial} A = -i\bar{\partial}\phi = -4i\bar{\partial}\log\alpha$

A NEW SOLUTION

[butti, grana, minasian, m.p., zaffaroni]

- There are also generic "solutions" (depending on the relative phases of a and b)
  - for $b = ae^{i\theta}$ $\exists$ a family of regular $\mathcal{N} = 1$
    - scalars (1): still complex

$$W_1 = H^{(1)} = F_3^{(1)}$$

    - tensors (6): deviation from self- duality of $G_3 = F_3 - \tau H$

$$2iab\, W_3 = e^{\phi}\, (a^2 + b^2)\, F_3^{(6)}$$
$$(a^2 - b^2)\, W_3 = -(a^2 + b^2)\, *_6\, H^{(6)}$$
$$2iab\, H^{(6)} = e^{\phi}\, (a^2 - b^2)\, *_6\, F_3^{(6)}$$

$\Rightarrow$ deformations of KS
interpolate between KS and MN

THE PT ANSATZ [papadopoulos,tseytlin]

- Ansatz for IIB solutions with fluxes
  - generalisation of the conifold metric
  - includes as special cases $\rightarrow$ singular and resolved conifold, KS and MN
- Metric: space-time of topology $\mathbf{R}^{1,3} \times \mathbf{R} \times S^2 \times S^3$

$$
\begin{aligned}
ds^2 \quad = \quad & e^{2A} dx_\mu^2 + e^{-6p-x} dt^2 + e^{-6p-x} \tilde{\epsilon}_3^2 \\
& + e^x (e^g + a^2 e^{-g})(e_1^2 + e_2^2) + e^{x-g} (\epsilon_1^2 + \epsilon_2^2) - 2\, a\, e^{x-g} (e_1\, \epsilon_1 + e_2\, \epsilon_2)
\end{aligned}
$$

- vielbeins

$$
\begin{aligned}
\tilde{\epsilon}_1 &= \epsilon_1 - a e_1 \\
\tilde{\epsilon}_2 &= \epsilon_2 - a e_2 \\
\tilde{\epsilon}_3 &= \epsilon_3 + \cos\theta_1 d\phi_1
\end{aligned}
$$

$$
\left.\begin{aligned}
e_1 &= d\theta_1 \\
e_2 &= -\sin\theta_1 d\phi_1
\end{aligned}\right\} \Longleftrightarrow S^2
\qquad
\left.\begin{aligned}
\epsilon_1 &= \sin\psi \sin\theta_2 d\phi_2 + \cos\psi d\theta_2 \\
\epsilon_2 &= \cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2 \\
\epsilon_3 &= d\psi + \cos\theta_2 d\phi_2 \\
d\epsilon_i &= -\tfrac{1}{2} \epsilon_{ijk} \epsilon_j \wedge \epsilon_k
\end{aligned}\right\} \Longleftrightarrow S^3
$$

- a, g, p, x, A $\rightarrow$ functions of the radial coordinate t only

- The fluxes authomatically satisfy the Bianchi identities

$$
\begin{aligned}
H &= h_2\,\tilde\epsilon_3 \wedge (\epsilon_1 \wedge e_1 + \epsilon_2 \wedge e_2) + dt \wedge \Big[h_1'(\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2) \\
&\quad + \chi'\,(e_1 \wedge e_2 - \epsilon_1 \wedge \epsilon_2) + h_2'\,(\epsilon_1 \wedge e_2 - \epsilon_2 \wedge e_1)\Big] \\
F_3 &= P\,\tilde\epsilon_3 \wedge \Big[\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2 - b\,(\epsilon_1 \wedge e_2 - \epsilon_2 \wedge e_1)\Big] + b'\,dt \wedge (\epsilon_1 \wedge e_1 + \epsilon_2 \wedge e_2) \\
F_5 &= \mathcal{F}_5 + \mathcal{F}_5^* \\
\mathcal{F}_5 &= K\,e_1 \wedge e_2 \wedge \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 .
\end{aligned}
$$

h_1, h_2, b , $K = Q + 2P[h_1 + bh_2]$, $\rightarrow$ functions of the radial coordinate only
$Q \sim$ number of regular D3, $P \sim$ number of fractional D3

- Symmetries
  - SU(2) $\times$ SU(2) isometry of the metric
  - $\mathbf{Z}_2$ action
    - exchange the two S^2 in the metric
    - on the fluxes it acts as

$$
F_3 \rightarrow -F_3 \qquad H_3 \rightarrow -H_3
$$

$$
F_5 \rightarrow F_5
$$

KLEBANOV-STRASSLER SOLUTION

- Near horizon geometry of N regular and M fractional D3 branes at the apex of a conifold $\rightarrow$ deformed conifold with fluxes
  - metric

$$A = -\frac{1}{4}\ln h \qquad a = -\frac{1}{\cosh t}$$

$$e^{6p+2x} = \frac{3}{2}\left(\coth t - t\,\mathrm{csch}^2 t\right) \qquad e^g = \tanh t$$

$$e^{2x} = \frac{(\sinh t\,\cosh t - t)^{2/3}}{16} h \qquad e^{\phi} = e^{\phi_0}$$

  - fluxes

$$h_1 = \frac{1}{2}(\coth t - t\coth^2 t) \qquad b = -\frac{t}{\sinh t}$$

$$h_2 = -\frac{(-1 + t\coth t)}{2\sinh t} \qquad \chi = 0$$

  - SU(3) structure $\quad \mathcal{A} = e^g, \quad \mathcal{B} = -a$
- The KS solution is $\mathbf{Z}_2$ invariant

MALDACENA-NUNEZ SOLUTION

• Near horizon geometry of N D5 branes wrapped on a 2-cycle $\to$ deformation of the linear dilaton background

• metric

$$a = -\frac{t}{\sinh t}\,, \qquad e^{2g} = -1 + 2t\coth t - \frac{t^2}{\sinh^2 t}$$

$$e^{2\phi} = e^{2\phi_0} e^{-g}\,\sinh t\,, \qquad e^{2A} = e^{2A_0} e^{-g/2}\,\sqrt{\sinh t}$$

$$e^{x} = e^{\phi_0} e^{g/2}\frac{\sqrt{\sinh t}}{2}\,, \qquad e^{6p} = \frac{4\,e^{-2\phi_0}}{\sinh t}$$

• fluxes $\qquad h_1 = h_2 = \chi = K = 0 \quad$ and $\quad a = b$

• SU(3) structure $\qquad \mathcal{A} = \coth t - t\,\mathrm{csch}^2 t \qquad \mathcal{B} = \mathrm{csch} t\sqrt{-1 + 2t\coth t - t^2\mathrm{csch}^2 t}$

• The MN solution is not $\mathbf{Z}_2$ invariant

THE INTERPOLATING SOLUTION

- Bianchi identities are authomatically satisfied $\Rightarrow$ look at the SUSY constraints
- Choose the SU(3) structure interpolating between the KS and MN ones

$$J = E_1 \wedge E_2 + E_3 \wedge E_4 + E_5 \wedge E_6$$
$$\Omega = (E_1 + iE_2) \wedge (E_3 + iE_4) \wedge (E_5 + iE_6)$$

with vielbeins E_i

$$E_1 = e^{\frac{x+g}{2}} e_1 \qquad E_2 = \mathcal{A} e^{\frac{x+g}{2}} e_2 - \mathcal{B} e^{\frac{x-g}{2}} \tilde{\epsilon}_2$$
$$E_3 = e^{\frac{x-g}{2}} \tilde{\epsilon}_1 \qquad E_4 = \mathcal{B} e^{\frac{x+g}{2}} e_2 - \mathcal{A} e^{\frac{x-g}{2}} \tilde{\epsilon}_2$$
$$E_5 = e^{-3p-\frac{x}{2}} dt \qquad E_6 = e^{-3p-\frac{x}{2}} \tilde{\epsilon}_3$$

$\mathcal{A}$ and $\mathcal{B}$ $\rightarrow$ functions of the radial direction ($\mathcal{A}^2 + \mathcal{B}^2 = 1$)

- Plug the PT ansatz in the SU(3) structure equations $\rightarrow$ set of differential equations for the functions A, a, p, x, g, h_1, h_2, χ, b, $\mathcal{A}$, $\mathcal{B}$, α, β
  - coupled differential equations for a and $v = e^{6p+2x}$
  - all the other functions depend on a and v

- We could not find an analytical solution $\longrightarrow$ numerical and power series analysis
  - IR asymptotic solution
    $\longrightarrow$ series expansion around $t = 0$
    $\longrightarrow$ impose the regularity of the solution at $t = 0$
  - UV asymptotic solution
    $\longrightarrow$ series expansion in $e^{-t/3}$ for $t \rightarrow \infty$
    $\longrightarrow$ impose the absence of an asymptotic Minkowskian region (near horizon geometry)
  - matching of the IR and UV parameters
    $\longrightarrow$ all the arbitrary constants are fixed in terms of ξ

$$a \rightarrow 1 - \xi\, t^2 + O(t^2) \qquad t \sim 0$$

$\longrightarrow$ numerical analysis shows regular solutions for

$$1/6 \leq \xi \leq 5/6$$

- ξ dependence of the solution

$$\xi = 1/2 \quad \longleftrightarrow \quad \text{KS solution}$$

$$1/6 < \xi \leq 1/2 \quad \longleftrightarrow \quad \text{solutions asymptotic to KS in the UV}$$

the dilaton is bounded

$$\xi = 1/6 \quad \longleftrightarrow \quad \text{MN solution}$$

$\Rightarrow$ ξ is the parameter of a family of regular solutions interpolating between KS and MN

- $\mathbf{Z}_2$ symmetry : for every solution (a, v) there is an another one with $(\tilde{a}, v)$
  - on the fields in the ansatz

$$\begin{aligned} \phi, h_i, x, p, A &\rightarrow \text{invariant} \\ \chi' &\leftrightarrow -\chi' \\ e^g(1+a^2) &\leftrightarrow e^{-g} \end{aligned}$$

  - on the parameter ξ: $\xi \leftrightarrow \xi - 1$

$\Rightarrow$ $\xi = 1/2$ (KS) is an invariant point

$\Rightarrow$ $1/6 \leq \xi < 1/2$ and $1/2 < \xi \leq 5/6$ are separated branches exchanged by $\mathbf{Z}_2$.

GAUGE THEORY INTERPRETATION

• Gauge theory dual of the KS solution $\rightarrow$ gauge theory on N regular and M fractional D3 branes at the apex of the conifold

$$\mathcal{N} = 1 \qquad SU(M+N) \times SU(N)$$

with chiral matter $A_i \in (N+M, \bar{N})$ and $B_j \in (\overline{N+M}, N)$ for $i, j = 1, 2$

• Symmetries

$SU(2) \times SU(2) \quad \rightarrow \quad$ rotation of A_i and B_j

$U(1)_B \quad \rightarrow \quad$ baryon number ($A_i \rightarrow e^{i\alpha} A_i$ and $B_i \rightarrow e^{-i\alpha} B_i$)

$U(1)_R \quad \rightarrow \quad$ anomalous chiral symmetry

• Seiberg dualities reducing the rank of the gauge groups $N \rightarrow N - M$. In the far IR

$$SU(M+p) \times SU(p) \qquad 0 \leq p \leq M$$

For $p \neq 0 \quad \rightarrow \quad$ D3 brane probes moving on the deformed conifold

$p = 0 \quad \rightarrow \quad$ regular SUGRA solution, KS and the interpolating solution ($F_5 \rightarrow 0$)

VACUA

- At the end of the cascade the gauge group is $SU(2M) \times SU(M)$ with $N_f = N_c \Rightarrow$
  - mesonic operators

$$N_{ij} = A_i B_j$$

  - baryonic operators

$$\mathcal{B} \sim \epsilon_{\alpha_1 \alpha_2 \ldots \alpha_{2M}} (A_1)_1^{\alpha_1} \ldots (A_1)_M^{\alpha_M} (A_2)_1^{\alpha_{M+1}} \ldots (A_2)_M^{\alpha_{2M}}$$
$$\bar{\mathcal{B}} \sim \epsilon^{\alpha_1 \alpha_2 \ldots \alpha_{2M}} (B_1)^1_{\alpha_1} \ldots (B_1)^M_{\alpha_M} (B_2)^1_{\alpha_{M+1}} \ldots (B_2)^M_{\alpha_{2M}}$$

- The effective superpotential has multiple vacua

$$W = \lambda (N_{ij})^\alpha_\beta \, (N_{kl})^\beta_\alpha \epsilon^{ik} \, \epsilon^{jl} + X \left(\det[(N_{ij})^\alpha_\beta] - \mathcal{B}\,\bar{\mathcal{B}} - \Lambda_{2M}^{4M} \right)$$

  - mesonic branch $\qquad \mathcal{B} = \bar{\mathcal{B}} = 0; \qquad \det[(N_{ij})^\alpha_\beta] = \Lambda_{2M}^{4M}$
  - baryonic branch $\qquad X = 0;\; N = 0; \qquad \mathcal{B} = \bar{\mathcal{B}} = i\Lambda_{2M}^{2M}$

- The baryonic branch preserves the $SU(2) \times SU(2)$ symmetry rotating A_i and B_j

KS is $SU(2) \times SU(2)$ invariant $\Rightarrow$ identify KS with a point of the baryonic branch

THE BARYONIC BRANCH FROM SUGRA

- The baryonic branch has complex dimension 1 and it is parametrized by ζ (Λ_{2M} is the UV scale of $SU(2M)$)

$$\mathcal{B} = i\zeta\Lambda_{2M}^{2M}, \qquad \bar{\mathcal{B}} = \frac{i}{\zeta}\Lambda_{2M}^{2M}$$

  - change in the phase of $\zeta \rightarrow$ breaking of $U(1)_B$
  - change in the value of $\zeta \rightarrow$ breaking of $\mathbf{Z}_2$ ($A_i \leftrightarrow B_i$ plus complex conjugation)

- The $U(1)_B$ is spontaneuosly broken by the baryon VEV's
  - $\exists$ a massless Goldstone boson
  - its SUGRA dual is a massless deformation of the RR 3-form [Gubser,Herzog,Klebanov]
  - $\mathcal{N} = 1$ SUSY $\Rightarrow \exists$ another real scalar mode
  - its SUGRA dual is the metric and NS deformation

- Our interpolating solution is dual to the flow along the baryonic branch
  - UV asymptotics ($r/r_0 \sim e^{t/3}$)

$$ds^2 \sim \frac{\text{const.}}{L^2} \frac{r^2}{\sqrt{\log r/r_0}} dx_m^2 + L^2 \frac{dr^2}{r^2} \sqrt{\log r/r_0} + L^2 \sqrt{\log r/r_0}\, ds^2_{T^{1,1}}$$

    - modification of the KS UV behaviour $L^2 = \frac{9}{\sqrt{2}} \frac{M\alpha'}{2} e^{\varphi_{UV}}$
    - family of vacua of $SU(N+M) \times SU(N)$ theory
  - k-string tension $\rightarrow$ fundamental strings at $t = 0$

$$T \sim \lambda \sin\psi \sqrt{1 + (\lambda^2 - 1)\cos^2\psi}$$

    - no universal IR behaviour
  - choice of the parameters
    - the baryonic branch corresponds to

$$\varphi_{UV} \qquad \text{fixed}, \; 1/6 < \xi \leq 1/2$$

$$\varphi_0 \quad \rightarrow \quad -\infty \qquad \left(e^{-2\,\phi_{UV}} = e^{-\phi_0} \sqrt{1 - \frac{9\,(1 - 2\,\xi)^2\,\lambda^2}{4}} \right)$$

    - the SUGRA solution is strongly coupled $\Rightarrow$ cannot describe the large VEV regime

IIB SOLUTIONS WITH SU(2) STRUCTURE

- SUGRA duals of deformations of $\mathcal{N} = 4$ Super Yang-Mills
  - mass terms for some of the adjoint fields
  - non delf-dual G_3
  - dielectric D3-branes (Myers effect)

- Spinor ansatz
  - SUSY projection for a dielectric D3-brane [pilch, warner]

$$\epsilon = \epsilon_1 + i\epsilon_2 = \Gamma_{0123}\,\epsilon + i\cos 2\phi\,\Gamma_{012345} * \epsilon$$

D3-brane D5-brane

  - this is an SU(2) structure [dall'agata]

PURE SPINOR INTERPRETATION

[minasian, m.p., zaffaroni in progress]

- Write the corresponding pure spinor equations
  - spinor choice

$$\begin{cases} \eta_+^1 = a\eta_+ + bz\eta_- \\ \eta_+^2 = x\eta_+ + yz\eta_- \end{cases} \qquad a = ix \qquad b = -iy$$

where $x = |x|e^{i\alpha}$, $y = |x|e^{i\beta}\tan\phi$, $|x|^2/\cos^2\phi = e^A$

  - pure spinors

$$\Phi_+ = \frac{1}{8}\Big[a\bar{x}e^{-ij} + b\bar{y}e^{ij} - i(a\bar{y}\omega + \bar{x}b\bar{\omega})\Big]e^{1/2z\bar{z}}$$

$$\Phi_- = \frac{1}{8}\Big[i(by\omega - ax\bar{\omega}) + (bxe^{ij} - aye^{-ij})\Big]z$$

- pure spinors equations
  - geometry

$$\mathrm{d}\Big(e^{3A}e^{i(\alpha+\beta)}\sin 2\phi z\Big) = 0 \qquad \text{conformally closed vector}$$

$$\mathrm{d}\Big[e^{2A}(j+\frac{i}{2}\cos 2\phi z\bar{z})\Big] = 0 \qquad \text{closed 2-form}$$

$$\mathrm{d}\Big(j^2 - \frac{1}{2}\hat{\omega}\bar{\hat{\omega}}\Big)\wedge z = 0 \qquad \text{SU(2) structure}$$

  - fluxes

$$e^{-4A}\mathrm{d}\Big(e^{4A}\cos 2\phi\Big) = - * F_5$$

$$e^{-4A}\mathrm{d}\Big(e^{4A}\sin 2\phi\, \mathrm{Im}\hat{\omega}\Big) = \cos 2\phi H - *F_3$$

$$\frac{i}{2}d\Big[e^{2A}\sin 2\phi \mathrm{Im}\hat{\omega} z\bar{z}\Big] - e^{2A}H\wedge(j+\frac{i}{2}\cos 2\phi z\bar{z}) = 0$$

$$e^{-4A}\mathrm{d}\Big(e^{4A}(\cos 2\phi\frac{j^2}{2}+\frac{i}{2}jz\bar{z})\Big) + \sin 2\phi H\wedge \mathrm{Im}\hat{\omega} = 0$$

$$\Big[\mathrm{d}\Big(\frac{1}{\sin 2\phi}(\cos^2\phi\hat{\omega}+\sin^2\phi\bar{\hat{\omega}})\Big) + iH\Big]\wedge z = 0$$

where $\hat{\omega} = e^{i(\alpha-\beta)}\omega$

EXAMPLES

- Known solutions: reinterpret a class of $\mathcal{N} = 1$ solutions with $U(1)^3$ isometry [warner and co.]. It includes
  - flow to the Leigh and Strassler fixed point [pilch,warner; gowdigere, warner]
  - flow from S_5/Z_2 to the conifold [halmagy, warner]
- In progress:
  - find solutions with fluxes dual to conformal field theories
  - Polchinski and Strassler solution?!

CONCLUSIONS

- Progress in understanding the geometrical properties of flux compactifications

$\longrightarrow$ G-structures

$\longrightarrow$ generalised complex geometry

- $\mathcal{N} = 1$ vacua are Generalised Calabi Yau manifolds
- Effective tool to interpret and find gravity duals of SUSY gauge theories
- Generalise the machinery of Calabi Yau compactifications
  - mirror symmetry proposal
  - branes in GCY
  - deformations of GCY manifolds
    $\longrightarrow$ moduli counting
    $\longrightarrow$ effective actions