# Twisted Conformal Symmetry in Noncommutative Two-Dimensional Quantum Field Theory

Belavin, Polyakov Zamolodchikov in Moyal sauce


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In this talk I will describe some aspects of the symmetry of noncommutative field theory that are gaining interest lately

In particular I will show how some symmetries which appeared to be broken by the noncommutative geometry are actually recovered as quantum (deformed) symmetries.

The symmetry I will discuss with more detail is two dimensional conformal symmetry

There is probably no need to motivate the choice of this symmetry:

It is an infinite dimensional Lie algebra, symmetry of the world sheet of strings

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We did it

Consider a two dimensional theory with Minkowski signature:

$$x^{\pm} = x^0 \pm x^1$$

A conformal transformation is just a purely "holomorphic" or "antiholomorphic" tranformation:

$$x^{\pm} \longrightarrow u^{\mp}(x^{\pm})$$

A treatment using $z = x^0 + ix^1, \bar{z} = x^0 - ix^1$ is also standard

The generators of this symmetry are:

$$\ell_n^{\pm} = -(x^{\pm})^{n+1}\partial_{\pm}$$

and generate the classical Virasoro-Witt algebra

$$[\ell_n^{\pm}, \ell_m^{\pm}] = (m-n)\ell_{n+m}^{\pm} \quad , \quad [\ell_n^{+}, \ell_m^{-}] = 0$$

One can write a conformal invariant action:

$$S = \int d^2x \; \partial_+\varphi\partial_-\varphi$$

The classical solutions are fields split in "left" and 'right" movers

$$\varphi = \varphi_+(x^+) + \varphi_-(x^-)$$

Quantizing the theory we have an expansion:

$$\phi(x^0, x^1) = \int_{-\infty}^{\infty} \frac{dk^1}{4\pi k_0} \left( a(k)e^{-ik^\mu x_\mu} + a^\dagger(k)e^{ik^\mu x_\mu} \right)$$

Using $k^0 = k_0 = |k^1|$ we have:

$$\phi(x^+, x^-) = \int_{-\infty}^{0} \frac{dk^1}{4\pi|k^1|} \left( a(k) e^{-i|k^1|x^+} + a^\dagger(k) e^{i|k^1|x^+} \right)$$

$$+ \int_{0}^{\infty} \frac{dk^1}{4\pi|k^1|} \left( a(k) e^{-ik^1 x^-} + a^\dagger(k) e^{ik^1 x^-} \right)$$

where reality of the fields implies $a(-k) = a^\dagger(k)$.

The canonical commutation relations give:

$$[a(p), a(q)] = 2p\delta(p+q)$$

The quantum currents $J^+(x) = \partial^+\phi, \quad J^-(x) = \partial^-\phi$ generate two commuting U(1) Kaç-Moody algebras with opposite central charges.

$$\left[J^\pm(x), J^\pm(y)\right] = -\frac{i}{\pi}\partial_\mp\delta(x^\mp - y^\mp)$$

This algebra gives the central extension of the Virasoro algebra

Quantum conformal invariance is proved showing that the components of the quantum stress-energy tensor generate the conformal algebra.

The quantum stress-energy tensor is the normal-ordered product

$$\Theta^{\pm\pm}(x) \propto : J^{\pm}(x) J^{\pm}(x) :$$

the other components vanish

normal ordering is defined:

$$: a(p)a(q) : = a(p)a(q) \text{ if } p < q$$

$$: a(p)a(q) : = a(q)a(p) \text{ if } p \geq q$$

The existence of Kaç-Moody quantum current algebras is a sufficient condition to ensure conformal invariance at the quantum level.

We can try now to put together conformal symmetry and Non-commutative Geometry.

*Unfortunately this is "impossible".*

We will consider the noncommutative plane equipped with the usual Moyal product

$$(f \star g)(x) = e^{\frac{i}{2}\theta(\partial_{x_0}\partial_{y_1} - \partial_{x_1}\partial_{y_0})} f(x) \cdot g(y)|_{x=y}$$

Where $\theta$ is a constant wit the dimensions of the square of a length

*Conformal theories are scale invariant and a theory with a fundamental scale cannot be scale invariant.*

*The Virasoro generators do not respect Leibnitz rule*

$$(x^{\pm})^{n+1}\partial_{\pm}(f \star g) \neq f \star (x^{\pm})^{n+1}\partial_{\pm} g + \Big((x^{\pm})^{n+1}\partial_{\pm} f\Big) \star g$$

or for that matter

$$(x^{\pm})^{n+1} \star \partial_{\pm}(f \star g) \neq f \star (x^{\pm})^{n+1} \star \partial_{\pm} g + \Big((x^{\pm})^{n+1} \star \partial_{\pm} f\Big) \star g$$

To say that a theory with a scale with the dimension of a length cannot be scale invariant is a little like saying that a theory with a constant of the dimensions of the angular momentum, $\hbar$, cannot be rotationally invariant. . .

We must understand how symmetries act on a non commutative space

The solution is to consider the symmetry to be a twisted quantum symmetry

(Wess and the Münich group Aschieri, Blohmann, Dimitriević, Meyer, Schupp, Chaichian-Kulish-Nishijima-Tureanu, Oeckl, Majid, Drinfeld . . . )

Consider the usual action of the Lie algebra $L$ of differential operators on the algebra $\mathcal{A}$ of functions with the usual commutative product

The usual product can be seen as a map from $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$

$m_0(f \otimes g) = fg$ pointwise multiplication

The Leibnitz rule impose a coalgebra structure of the Lie algebra:

$$\ell(fg) = \ell(f)g + f\ell(g) = m_0(\Delta(L)(f \otimes g))$$

where

$\Delta : L \to L \otimes L$

$$\Delta(\ell) = \ell \otimes 1 + 1 \otimes \ell$$

Consider the Moyal product as follows

$$(f \star g)(x) = m_0[\mathcal{F}^{-1} f \otimes g] \equiv m_\theta[f \otimes g]$$

where $m_0(f \otimes g) = fg$ is the ordinary product and

$$\mathcal{F} = e^{\frac{i}{2}\theta^{\mu\nu}\partial_{x_\mu}\otimes\partial_{y_\nu}} = e^{\frac{i}{2}\theta(\partial_{x_0}\otimes\partial_{y_1}-\partial_{x_1}\otimes\partial_{y_0})}$$

is called the twist.

The Noncommutative plane is obtained twisting the tensor product, and using the ordinary product.

We have to then revise the Leibnitz rule:

$$\ell(f \star g) = m_\theta \Delta_\theta (f \otimes g) = m_0 \Delta \ell(\mathcal{F}^{-1}(f \otimes g))$$

where

$$\Delta_\theta = \mathcal{F} \Delta \mathcal{F}^{-1}$$

The algebra structure remains unchanged, what changes is the colagebra structure, that is the way to "put together representations".

counit and antipode remain unchanged.

The new coproduct is a deformation of the old one:

$$\Delta_\theta(\ell_n^+) = \mathcal{F}^{-1}\Delta_0(\ell)\mathcal{F} =$$
$$= (\mathbf{1}\otimes x^+ - \theta\partial_- \otimes \mathbf{1})^{n+1}(-\mathbf{1}\otimes\partial_+) + (x^+\otimes\mathbf{1} - \mathbf{1}\otimes\theta\partial_-)^{n+1}(-\partial_+\otimes\mathbf{1})$$

and analogously:

$$\Delta_\theta(\ell_n^-) = (-\mathbf{1}\otimes\partial_-)(\mathbf{1}\otimes x^- - \theta\partial_+\otimes\mathbf{1})^{n+1} + (-\partial_-\otimes\mathbf{1})(x^-\otimes\mathbf{1} - \mathbf{1}\otimes\theta\partial_+)^{n+1}$$

Conformal symmetry survives, but in a twisted form.

What happens to the quantum theory in the presence of this twisted symmetry?

Consider the Kaç-Moody algebra in this twisted setting.

$$\left[J^{\pm}(x), J^{\pm}(y)\right]_{\star}, \quad \left[J_{+}(x), J_{-}(y)\right]_{\star}$$

Combine fields at different points using the twist, so that, with a slight abuse of notation

$$f(x) * f(y) = m_0 \left[\mathcal{F}(f \otimes g)\right](x, y) = e^{\frac{i}{2}\theta^{\mu\nu}\partial_{x_\mu}\partial_{y_\nu}} f(x) f(y)$$

The deformation in the product does not affect the commutators in the same sector

$$\left[J^{\pm}(x), J^{\pm}(y)\right]_{\star} = -\frac{i}{\pi}\partial_{\mp}\delta(x^{\mp} - y^{\mp})$$

because

$$e^{\frac{i}{2}\partial_x \wedge \partial_y} e^{\mp ikx^{\pm}} e^{\mp ipy^{\pm}} = e^{i\theta(\partial_{x^-}\partial_{y^+} - \partial_{x^+}\partial_{y^-})} e^{\mp ikx^{\pm}} e^{\mp ipy^{\pm}} = e^{\mp ikx^{\pm}} e^{\mp ipy^{\pm}}$$

with

$$\partial_x \wedge \partial_y = \theta^{\mu\nu}\partial_{x_\mu}\partial_{y_\nu}$$

In each chiral sector the symmetry is unchanged. This is to be expected since the $\star$-product between two functions of $x^+$ or $x^-$ alone is the same as the usual product. Likewise the coproduct when acting on pairs of such functions is the same as the undeformed one

The effect of the noncommutativity of the plane is only felt when $x^+$ and $x^-$ are put together in:

$$\left[J^+(x), J^-(y)\right]_\star$$

To calculate it we have to use:

$$e^{\frac{i}{2}\partial_x \wedge \partial_y} e^{-ikx^+} e^{ipy^-} = e^{-i\theta kp} e^{-ikx^+} e^{ipy^-}$$

Using the usual commutation relations for the $a(k)$ we would not obtain commuting currents and the theory would not be conformally invariant.

The coupling between the left and right sectors inhibits the cancelation of some terms.

Can we fix this?

It is in fact possible to still obtain the proper algebra with a deformation of the the quantum commutation relations

$$a(p)a(q) = \mathcal{F}^{-1}(p,q)a(q)a(p) + 2p\delta(p+q)$$

where we used the inverse of the twist in momentum space

$$\mathcal{F}^{-1}(p,q) = e^{-\frac{i}{2}p\wedge q} = e^{-i\theta(|p|q-|q|p)}$$

Notice that the action of $\mathcal{F}^{-1}(p,q)$ is always zero when considering the commutator between currents of the same chirality because $p$ and $q$ have the same sign

The quantum twist compensates the one in spacetime, and the current algebra becomes the old one

Thanks to the new twisted commutation relation between the quantum creation and annihilation operators the quantum theory is (twisted) conformally invariant.

Thanks to relations left and right currents commute, while currents of the same chirality yield a central term, as in the standard theory:

$$\left[J^{\pm}(x), J^{\pm}(y)\right]_{\star} = -\frac{i}{\pi}\partial_{\mp}\delta(x^{\mp} - y^{\mp}), \quad \left[J^{+}(x), J^{-}(y)\right]_{\star} = 0$$

These twisted, "quantum-plane-like" commutation relations among the $a$'s are not new, they appeared in higher dimensional theories (Balachandran, Mangano, Pinzul, Vaidya)

# Conclusions and Perspectives

We have constructed a quantum conformal noncommutative field theory

There are several applications of conformal field theory in physics. The theory of strings is the most famous one, but there are also applications in solid state. How is the twist in the commutation relations going to change things?

Likewise the mathematical structure underlying a conformal theory is extremely rich. The whole theory of Vertex Operator Algebras is based on it. The new coproduct is changing the way representations are put together. What structure is the new coproduct putting on the new vertices? on the representations of the algebra? Are they known?