"Nonlinear sigma model: symmetric renormalization of the perturbative series"

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- Introduction. Problem: give a meaning to the perturbative expansion of the nonlinear sigma model in 4 dimensions.

$$S = \frac{1}{2}\int d^Dx \left(\partial^\mu\phi_a\partial_\mu\phi_a + g^2\frac{\phi_a\partial^\mu\phi_a\phi_b\partial_\mu\phi_b}{\phi_0^2}\right)$$
$$= \frac{1}{2}\int d^Dx \left(\partial^\mu\phi_a\partial_\mu\phi_a + \frac{1}{g^2}\partial^\mu\phi_0\partial_\mu\phi_0\right)$$

with the constraint

$$\phi_0^2 + g^2\vec{\phi}^2 = m_D^2, \qquad m_D \equiv m^{\frac{D}{2}-1}.$$

Group element of SU(2)

$$\Omega = \frac{1}{m_D}\left[\phi_0 + ig\tau_a\phi_a\right] \qquad t_a = \frac{\tau_a}{2}.$$

Content (i.e. conclusions)

- Flat connection. The sigma model is reformulated in terms of the flat connection $(SU(2))$

$$F_\mu \equiv \Omega \partial_\mu \Omega^\dagger, \qquad \Omega = \frac{1}{m_D}\left[\phi_0 + ig^2 \tau_a \phi_a\right].$$

  The constraint $\Omega^\dagger \Omega = 1$ is described by the external source $K_0 \phi_0$.

- Naïve perturbation series is a solution in D-dimensions.

  The defining functional equation has a perturbative expansion (in loops) in D dimensions. It depends only on one parameter. In the limit $D = 4$ divergences show up and a subtraction is need. In the minimal subtraction a further scale can be introduced.(Use of Quantum Action Principle!)

- Hierarchy

  The defining functional equation has a strong hierarchal structure, that allows to evaluate all the amplitudes in terms of those involving the composite operators (flat connection and constraint). This yields a weak power counting theorem that reduces to a finite number of the independent divergent amplitudes at each order (at each order the number of divergent amplitudes is divergent). Then at each order of the loop expansion there is a finite number of arbitrary parameters. Ex. at 1-loop there are 7 free parameters. A phenomenological use of the nonlinear sigma model is then possible and consistent.

- Symmetric subtraction of infinities

  Once the power counting theorem is not valid (due to nonrenormalizability), a symmetric regularization does not guarantee that the theory is stable under subtraction of infinities. In the nonlinear sigma model the defining equations are stable under subtraction of the poles in D=4 (we have some formal proof and an explicit example at the two-loop level).

- To learn how to deal with a non renormalizable field theory. Phenomenology?

  Minimal subtraction allows a formulation of a theory in terms of two parameters. This is a well defined theory that can be falsified. The other extreme attitude is to allow all possible free parameters at a given number of loops (7 at one loop). The theory maintains a predictive power.

- Limits on the linear sigma model.

  Once a well-definite theory is given for the nonlinear sigma model is given, one can study the limits for the linear model: the chiral Lagrangian limit (momenta $<< m_\sigma$) and the limit of $M_H \to \infty$. In the first case we have perfect matching for the leading terms, while in the second we need final tuning in order to remove the $\ln M_H$.

- Plan.
  - Previous work
  - Which quantization?
  - Flat connection
  - Functional equation approach
  - Hierarchy: how to compute graphs
  - Limit D=4: stability under subtraction
  - Phenomenological applications
  - Limits on the linear sigma model.
- Global counter terms for ϕ^4 amplitude.

  The symmetric global counterterms

$$tr\ [\partial_\mu\Omega^\dagger\partial^\mu\Omega]\ tr\ [\partial_\nu\Omega^\dagger\partial^\nu\Omega], \quad tr\ [\partial_\mu\Omega^\dagger\partial^\nu\Omega]\ tr\ [\partial^\mu\Omega^\dagger\partial_\nu\Omega], \quad tr\ [\Box\Omega^\dagger\Box\Omega] \qquad (1)$$

  cannot renormalized all the divergences, due to the presence of a non trivial Haar measure in the path integral.

- Renormalization? A possibility via field redefinition $\phi_a \to f(\vec{\phi}^2)\phi_a$? Tātaru (1975) discovers 1-loop divergences in $\phi - \phi$ scattering that cannot be written as counterterms in eq. (1). Appelquist and Bernard (1981) show that these terms can be absorbed by a field redefinition if space-time derivatives are admitted.

How to quantize?

- Eulero-Lagrange? $\phi_a \to \phi_a + \delta\phi_a$

$$\int \mathcal{D}[\vec{\phi}] \exp i(S + \int d^D z K_0(z)\phi_0(z)) \left((\frac{1}{\phi_0}\Box\phi_0\phi_a - \Box\phi_a) + K_a - g^2 \frac{1}{\phi_0} K_0 \phi_a \right) = 0$$

- Path integral with group measure? One has a left multiplication invariance

$$\Omega \to U(\delta\omega)\Omega \sim (1 + i\frac{g}{2}\tau_a \delta\omega_a)\Omega$$

or

$$\delta\phi_0 = -g^2 \frac{\delta\omega_a}{2}\phi_a$$
$$\delta\phi_a = \frac{\delta\omega_a}{2}\phi_0 + g\frac{\delta\omega_c}{2}\epsilon_{abc}\phi_b.$$

- Then we need an invariant measure (Haar) in the path integral:

$$d^4\phi\,\delta(-m_D^2+\vec{\phi}^2+\phi_0^2)=\frac{d^3\phi}{2\phi_0}$$

This measure is not invariant under $\phi_a \to \phi_a+\delta\phi_a \quad \Longrightarrow$ Eulero - Lagrange equations are sick.

- Instead consider the flat connection

$$F_{a\mu}=Tr\Big[\tau_a F_\mu\Big]=Tr\Big[\tau_a\frac{i}{g}\Omega\partial_\mu\Omega^\dagger\Big]=\frac{2}{m_D^2}\Big[(\phi_0\partial_\mu\phi_a-\partial_\mu\phi_0\phi_a)+g\epsilon_{abc}\partial_\mu\phi_b\phi_c\Big]$$

$$\frac{m_D^2}{8}F_a^\mu F_{a\mu}=\frac{m_D^2}{4g^2}Tr\Big[\partial^\mu\Omega\partial_\mu\Omega^\dagger\Big]=\frac{1}{2}\Big[\partial^\mu\phi_a\partial_\mu\phi_a+\frac{1}{g^2}\partial^\mu\phi_0\partial_\mu\phi_0\Big]$$

and the vertex functional

$$\Gamma^{(0)}\equiv\int d^Dx\Big\{\frac{1}{2}\left(\partial^\mu\phi_a\partial_\mu\phi_a+\frac{1}{g^2}\partial^\mu\phi_0\partial_\mu\phi_0\right)+K_0\phi_0+J_a^\mu F_{a\mu}\Big\}.$$

- Use the functional equation that comes from infinitesimal transformations that leave invariant the measure:

$$-\frac{m_D^2}{2}\partial^\mu\frac{\delta W}{\delta J_a^\mu} - g^2\frac{\delta W}{\delta K_a}K_0 + \frac{\delta W}{\delta K_0}K_a + g\epsilon_{abc}K_b\frac{\delta W}{\delta K_c} - 2\mathcal{D}_{ab\mu}[\frac{\delta W}{\delta J}]J_b^\mu = 0.$$

For the generating functional of the 1PI amplitudes one has

$$\frac{m_D^2}{2}\partial^\mu\frac{\delta\Gamma}{\delta J_a^\mu} + g^2\phi_a K_0 + \frac{\delta\Gamma}{\delta K_0}\frac{\delta\Gamma}{\delta\phi_a} + g\epsilon_{abc}\frac{\delta\Gamma}{\delta\phi_b}\phi_c + 2\mathcal{D}[\frac{\delta\Gamma}{\delta J}]_{ab}^\mu J_{b\mu} = 0. \qquad (2)$$

- Feynman rules. The Feynman rules in D dimensions are provide by $\Gamma^{(0)}$ which satisfies eq. (2). The amplitudes obtained in this way do satisfy eq. (2)!!, provided tadpoles are omitted

$$\int d^Dk\frac{1}{(k^2)^\alpha} = 0, \qquad \alpha\in\mathcal{C}.$$

$D = 4$ or Renormalization

- For $D = 4$ the amplitudes are singular. Subtraction procedure by removing the poles in the Laurent expansion of the normalized amplitudes

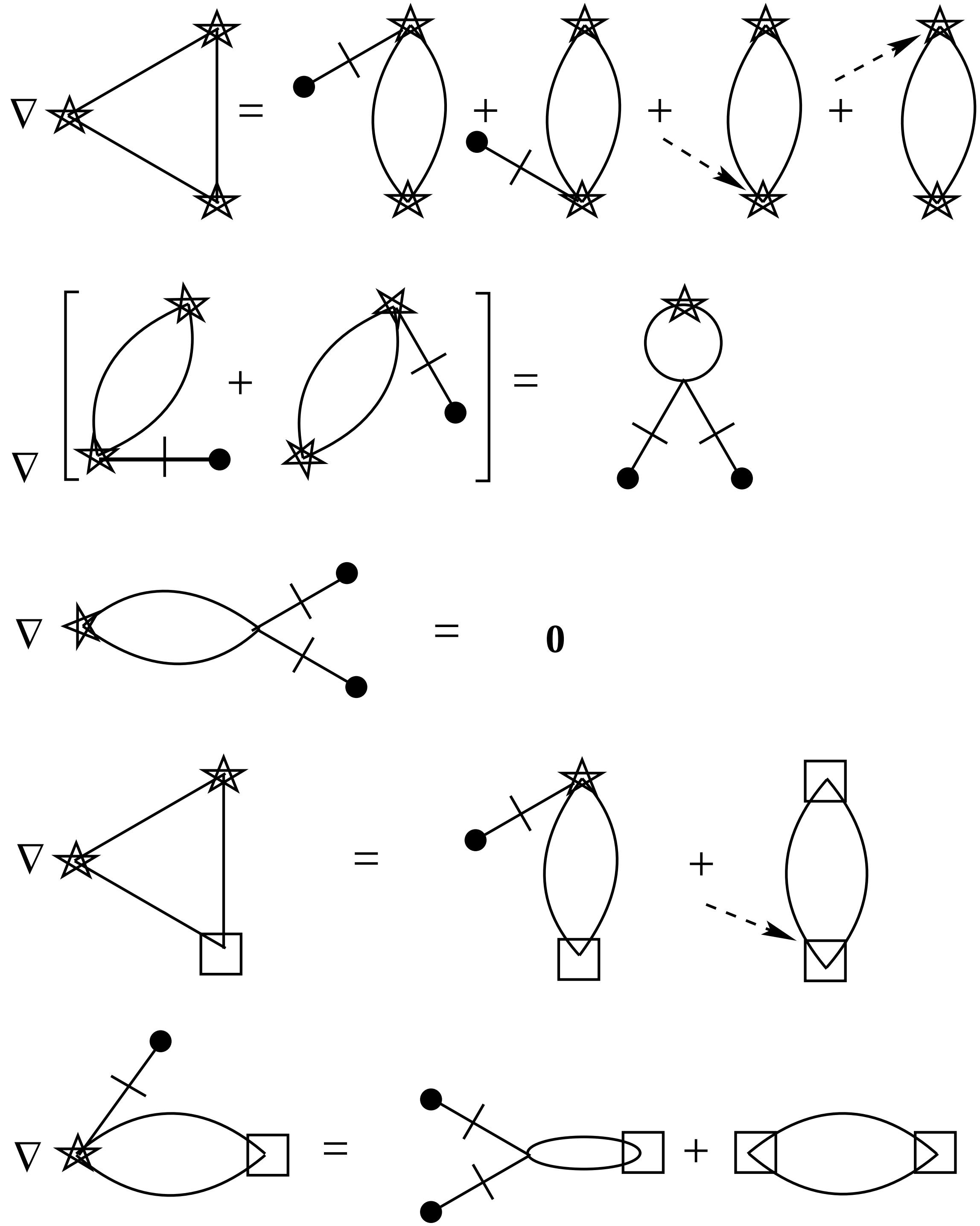
$$\left(\frac{m_D}{m}\right)^{2(n-1)} \Gamma_{J^{\mu_1}_{a_1} \dots J^{\mu_n}_{a_n}} = m^{(n-1)(D-4)} \Gamma_{J^{\mu_1}_{a_1} \dots J^{\mu_n}_{a_n}}.$$

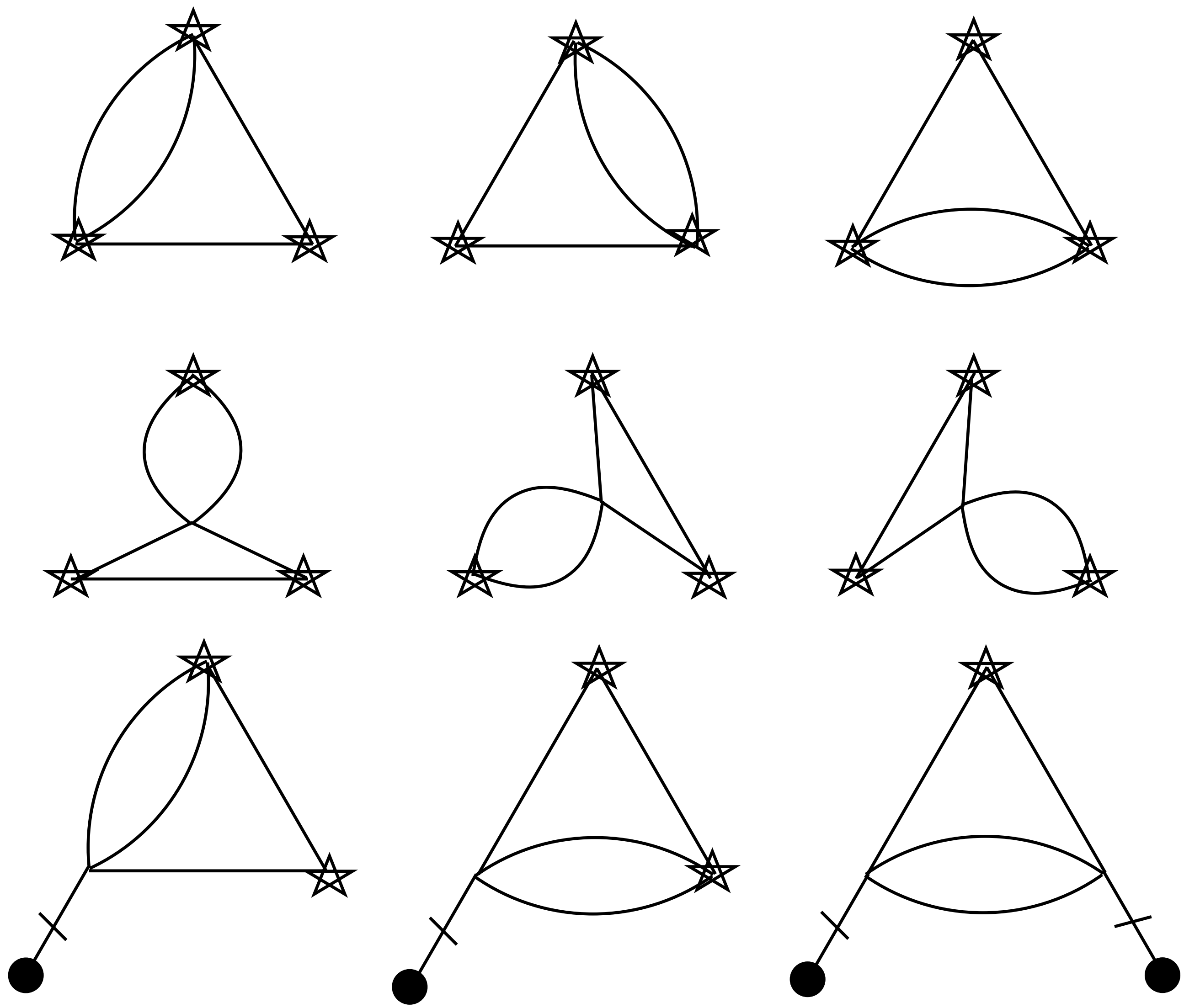
- Suppose that the poles in D=4 have been removed up to order n-1, then

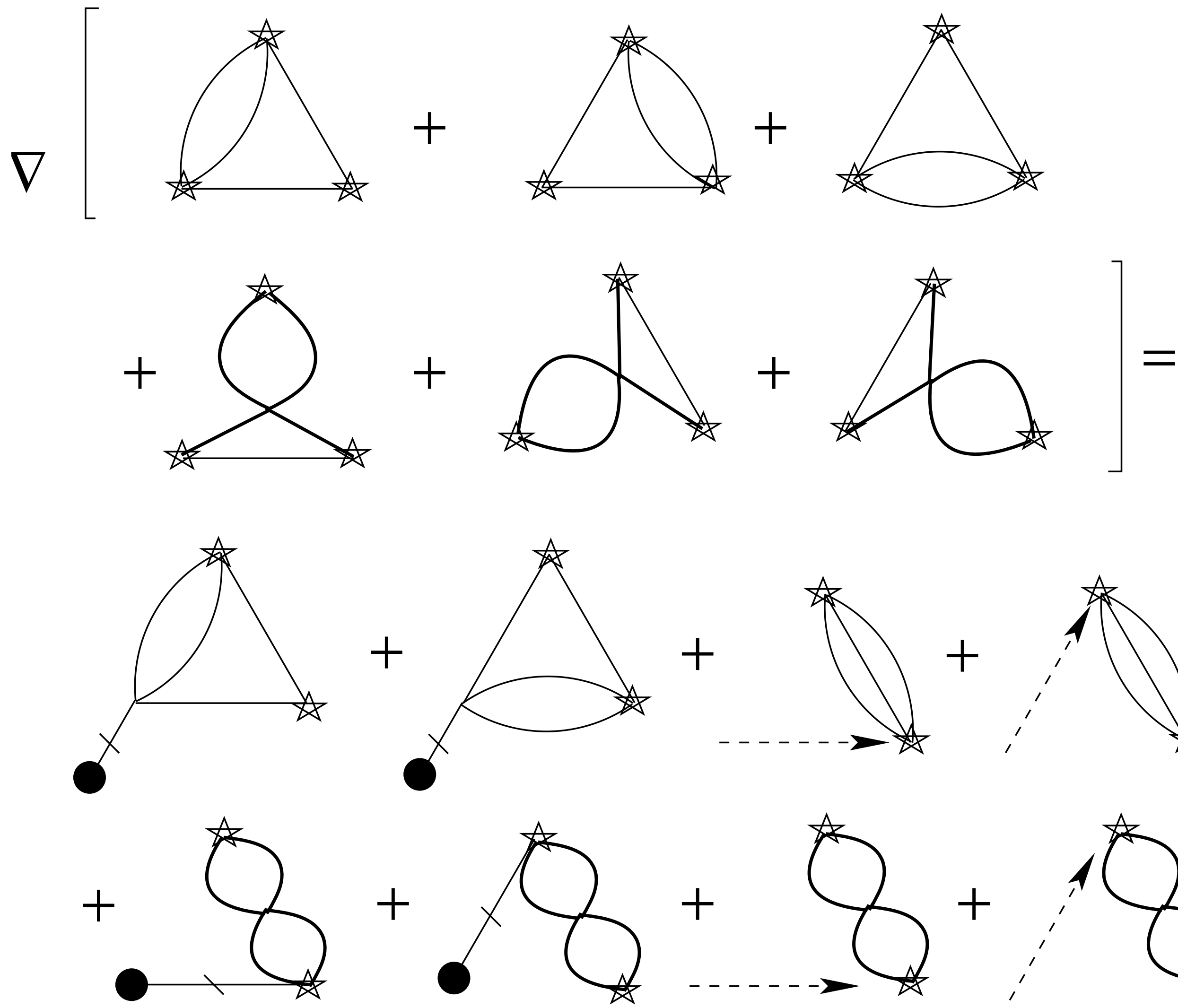
$$\Big(\frac{m_D^2}{2}\partial^\mu \frac{\delta}{\delta J^\mu_a} + \frac{\delta\Gamma^{(0)}}{\delta\phi_a}\frac{\delta}{\delta K_0} + \frac{\delta\Gamma^{(0)}}{\delta K_0}\frac{\delta}{\delta\phi_a} + g\epsilon_{abc}\phi_c\frac{\delta}{\delta\phi_b} - 2g\epsilon_{abc}J_{b\mu}\frac{\delta}{\delta J^\mu_c}\Big)\Gamma^{(n)}$$
$$+ \sum_{j=1,n-1} \frac{\delta\Gamma^{(j)}}{\delta K_0}\frac{\delta\Gamma^{(n-j)}}{\delta\phi_a} = \Delta^{(n)} \qquad (3)$$

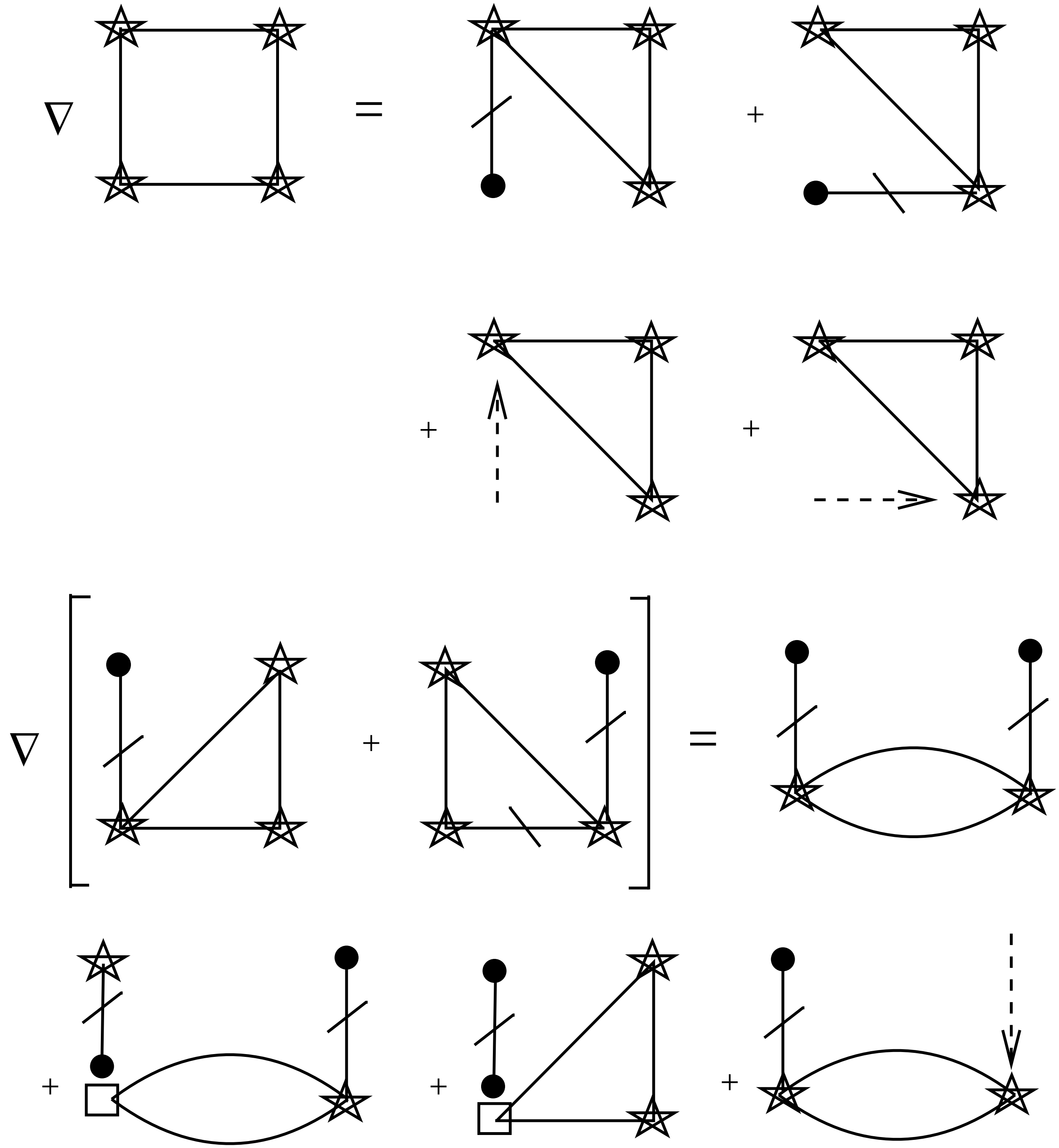
If the subtraction procedure removes $\Delta^{(n)}$ then, the renormalization program is consistent with the functional equation. Free parameters are associated to the solutions of the homogeneous equation (3).

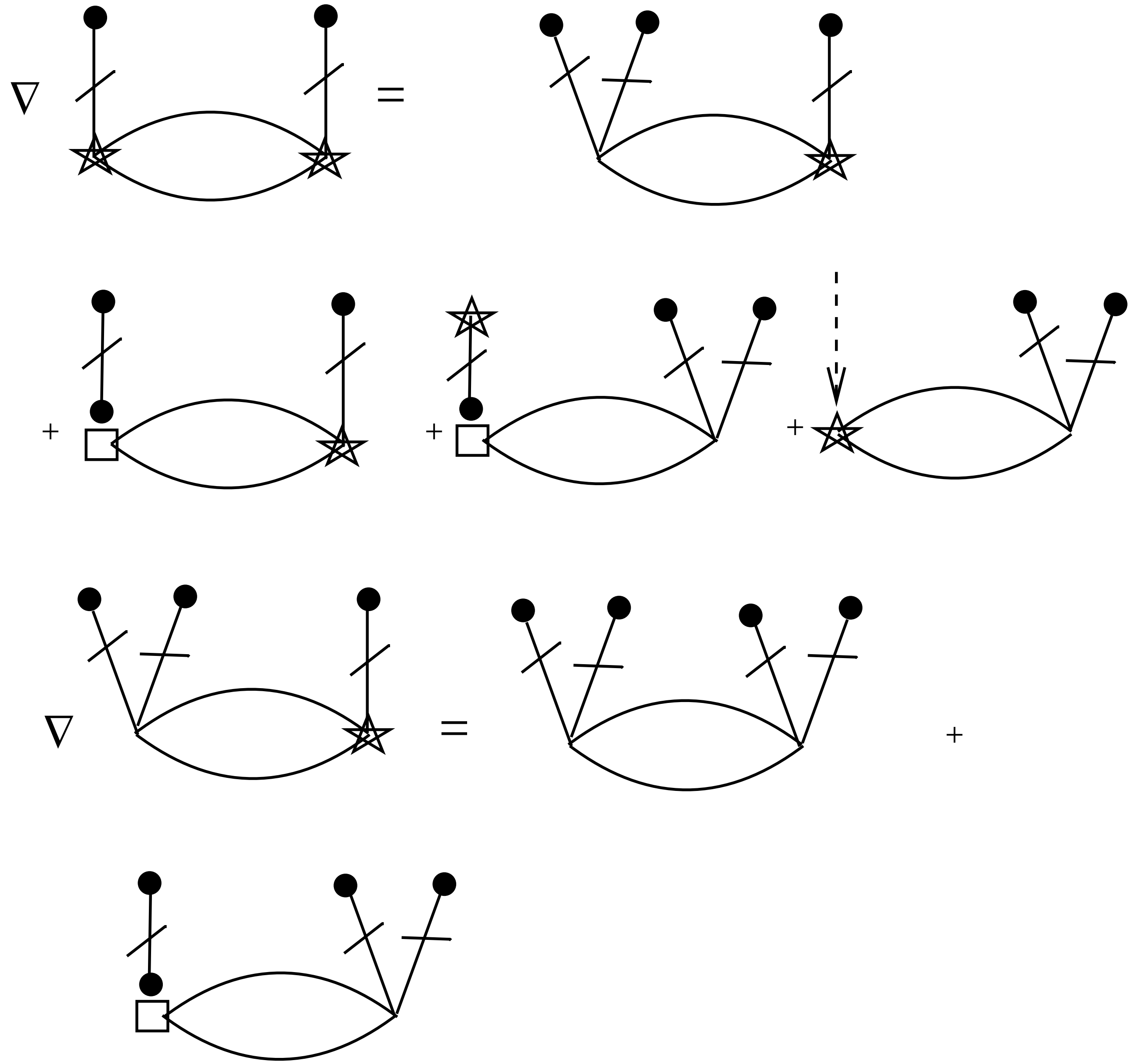
- Hierarchy! Start from amplitudes with the highest number of composite operators (J and K_0 external currents) and from them derive the amplitudes involving also the $\vec{\phi}$ fields.











Weak Power Counting Theorem

- Superficial convergence if

$$N_J + 2N_{K_0} > (D-2)n + 2\,,$$

where N_J, N_{K_0} are the number of F and ϕ_0 insertions (no ϕ external legs) and n is the number of loops. Ex: at one loop graphs with $N_J > 4$ are convergent the same for $N_{K_0} > 2$.

- Construct the counterterms. Find all local solutions of the homogeneous eq. (3). They are constructed by using the invariants

$$\overline{K}_0 = \frac{m_D^2 K_0}{\phi_0} - \phi_a \frac{\delta S_0}{\delta \phi_a}$$

where

$$S_0 = \frac{m_D^2}{8} \int d^4x \left(F_{a\mu} + \frac{4}{m_D^2} J_{a\mu} \right)^2$$

and their ordinary derivatives, the undifferentiated Ω and the combination

$$F_a^\mu + \frac{4}{m_D^2} J_a^\mu$$

and its subsequent covariant derivatives w.r.t. F_μ.

One-loop, D=4

At the one-loop level we divergent amplitudes only for $\Gamma_{JJJJ}, \Gamma_{K_0K_0}, \Gamma_{JJJ}, \Gamma_{JJK_0}, \Gamma_{JJ}$. The possible invariants are

$$\mathcal{I}_1 = \int d^Dx \left[D_\mu(F + \frac{4}{m_D^2}J)_\nu\right]_a \left[D^\mu(F + \frac{4}{m_D^2}J)^\nu\right]_a ,$$

$$\mathcal{I}_2 = \int d^Dx \left[D_\mu(F + \frac{4}{m_D^2}J)^\mu\right]_a \left[D_\nu(F + \frac{4}{m_D^2}J)^\nu\right]_a ,$$

$$\mathcal{I}_3 = \int d^Dx \, \epsilon_{abc} \left[D_\mu(F + \frac{4}{m_D^2}J)_\nu\right]_a \left(F_b^\mu + \frac{4}{m_D^2}J_b^\mu\right)\left(F_c^\nu + \frac{4}{m_D^2}J_c^\nu\right) ,$$

$$\mathcal{I}_4 = \int d^Dx \left(\frac{m_D^2 K_0}{\phi_0} - \phi_a \frac{\delta S_0}{\delta \phi_a}\right)^2 ,$$

$$\mathcal{I}_5 = \int d^Dx \left(\frac{m_D^2 K_0}{\phi_0} - \phi_a \frac{\delta S_0}{\delta \phi_a}\right)\left(F_b^\mu + \frac{4}{m_D^2}J_b^\mu\right)^2 ,$$

$$\mathcal{I}_6 = \int d^Dx \left(F_a^\mu + \frac{4}{m_D^2}J_a^\mu\right)^2 \left(F_b^\nu + \frac{4}{m_D^2}J_b^\nu\right)^2 ,$$

$$\mathcal{I}_7 = \int d^Dx \left(F_a^\mu + \frac{4}{m_D^2}J_a^\mu\right)\left(F_a^\nu + \frac{4}{m_D^2}J_a^\nu\right)\left(F_{b\mu} + \frac{4}{m_D^2}J_{b\mu}\right)\left(F_{b\nu} + \frac{4}{m_D^2}J_{b\nu}\right) .$$

The explicit evaluation of the pole parts gives

$$
\begin{aligned}
\widehat{\Gamma}^{(1)} \;&=\; \frac{1}{D-4}\Big[-\frac{1}{12}\frac{g^2}{(4\pi)^2}\frac{m_D^2}{m^2}\Big(\mathcal{I}_1-\mathcal{I}_2-g\mathcal{I}_3\Big)+\frac{1}{(4\pi)^2}\frac{g^4}{48}\frac{m_D^2}{m^2}\Big(\mathcal{I}_6+2\mathcal{I}_7\Big) \\
&+\frac{1}{(4\pi)^2}\frac{3}{2}\frac{g^4}{m^2m_D^2}\mathcal{I}_4+\frac{1}{(4\pi)^2}\frac{1}{2}\frac{g^4}{m^2}\mathcal{I}_5\Big] \,.
\end{aligned}
$$

As an example we can get the counterterm for the four-point function by projecting $\widehat{\Gamma}^{(1)}$ in eq.(4) on the monomials involving ϕ_a, K_0 and $J_{a\mu}$. By direct computation the projection of the combination $\mathcal{I}_1-\mathcal{I}_2-g\mathcal{I}_3$ on the relevant monomials is zero, while the contribution from $\mathcal{I}_6+2\mathcal{I}_7$ and $\mathcal{I}_4$, $\mathcal{I}_5$ gives rise to

$$
\begin{aligned}
\widehat{\Gamma}^{(1)}[\phi\phi\phi\phi] \;&=\; -\frac{1}{D-4}\frac{g^4}{m_D^2m^2(4\pi)^2} \\
&\int d^Dx\,\Big[-\frac{1}{3}\partial_\mu\phi_a\partial^\mu\phi_a\partial_\nu\phi_b\partial^\nu\phi_b-\frac{2}{3}\partial_\mu\phi_a\partial_\nu\phi_a\partial^\mu\phi_b\partial^\nu\phi_b \\
&-\frac{3}{2}\phi_a\Box\phi_a\phi_b\Box\phi_b-2\phi_a\Box\phi_a\partial_\mu\phi_b\partial^\mu\phi_b\Big] \,.
\end{aligned}
$$

Stability under subtraction

- After the pole subtraction, is the equation still valid?

  The explicit evaluation of the pole parts at one loop gives

$$\Big(\frac{m_D^2}{2}\partial^\mu\frac{\delta}{\delta J_a^\mu}+\frac{\delta\Gamma^{(0)}}{\delta\phi_a}\frac{\delta}{\delta K_0}+\frac{\delta\Gamma^{(0)}}{\delta K_0}\frac{\delta}{\delta\phi_a}+g\epsilon_{abc}\phi_c\frac{\delta}{\delta\phi_b}-2g\epsilon_{abc}J_{b\mu}\frac{\delta}{\delta J_c^\mu}\Big)\Gamma^{(1)}\Big|_{\rm POLE}=0.$$

  The counterterm

$$\widehat{\Gamma}^{(1)}\equiv-\Gamma^{(1)}\Big|_{\rm POLE}$$

  does not spoil the functional equation!

- At two loop the nonlinear term is dangerous

  After the introduction of the one-loop counterterm

$$\Big(\frac{m_D^2}{2}\partial^\mu\frac{\delta}{\delta J_a^\mu}+\frac{\delta\Gamma^{(0)}}{\delta\phi_a}\frac{\delta}{\delta K_0}+\frac{\delta\Gamma^{(0)}}{\delta K_0}\frac{\delta}{\delta\phi_a}+g\epsilon_{abc}\phi_c\frac{\delta}{\delta\phi_b}-2g\epsilon_{abc}J_{b\mu}\frac{\delta}{\delta J_c^\mu}\Big)\Gamma^{(2)}$$
$$+\frac{\delta\Gamma^{(1)}}{\delta K_0}\frac{\delta\Gamma^{(1)}}{\delta\phi_a}=\Delta^{(2)}$$

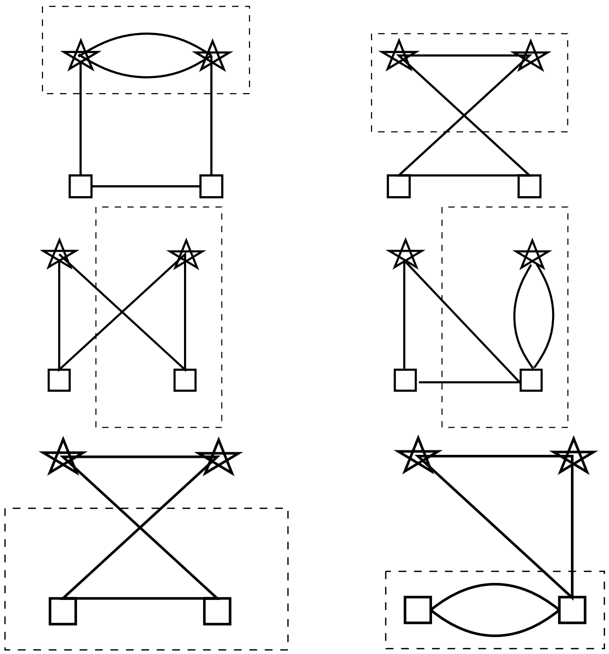
  and then

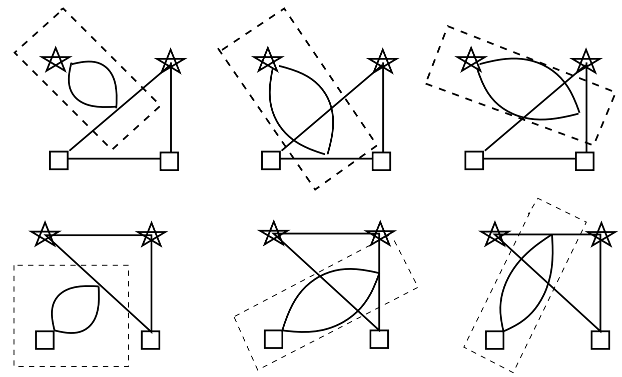
$$\Big(\frac{m_D^2}{2}\partial^\mu\frac{\delta}{\delta J_a^\mu}+\frac{\delta\Gamma^{(0)}}{\delta\phi_a}\frac{\delta}{\delta K_0}+\frac{\delta\Gamma^{(0)}}{\delta K_0}\frac{\delta}{\delta\phi_a}+g\epsilon_{abc}\phi_c\frac{\delta}{\delta\phi_b}-2g\epsilon_{abc}J_{b\mu}\frac{\delta}{\delta J_c^\mu}\Big)\Gamma^{(2)}\Big|_{\rm POLE}=\Delta^{(2)}$$

- Subtraction of the pole must remove $\Delta^{(2)}$ at the same time!.

A non trivial case

$$\frac{m_D^2}{2}\partial^\mu \Gamma^{(2)}_{J_a^\mu J K_0 K_0} + m_D \Gamma^{(2)}_{\phi_a J K_0 K_0} + \Gamma^{(2)}_{K_0 K_0 K_0}\Gamma^{(0)}_{\phi_a J}$$
$$+\Gamma^{(1)}_{K_0 K_0}\Gamma^{(1)}_{\phi_a K_0 J} = \Delta^{(2)}.$$





Subtraction of the poles for

$$\left(\frac{m_D}{m}\right)^4 \Gamma_{2K_0 2J}, \qquad \left(\frac{m_D}{m}\right) \Gamma_{3K_0}$$

subtracts also $\Delta^{(2)}$ completely.

Solution of the linearized equation

Introduce the rescaled background current

$$\widehat{J}_{a\mu} \equiv -\frac{4}{m_D^2} J_{a\mu} \,.$$

Use the *ad hoc* BRS transformations in terms of classical Grassmannian local parameters $\omega_a(x)$

$$s\widehat{J}_{a\mu} = \partial_\mu \omega_a + g\epsilon_{abc}\widehat{J}_{b\mu}\omega_c \,, \qquad s\phi_a = \frac{1}{2}\omega_a\phi_0 + \frac{1}{2}g\epsilon_{abc}\phi_b\omega_c \,, \qquad sK_0 = \frac{1}{2}\omega_a \frac{\delta\Gamma^{(0)}}{\delta\phi_a} \,.$$

$$s^2 = 0 \qquad \text{if} \qquad s\omega_a = \frac{1}{2}g\epsilon_{abc}\omega_b\omega_c.$$

Then the linearized functional equation (for the free parameters $\Delta\Gamma^{(n)}$)

$$\Big(\frac{m_D^2}{2}\partial^\mu \frac{\delta}{\delta J_a^\mu} + \frac{\delta\Gamma^{(0)}}{\delta\phi_a}\frac{\delta}{\delta K_0} + \frac{\delta\Gamma^{(0)}}{\delta K_0}\frac{\delta}{\delta\phi_a} + g\epsilon_{abc}\phi_c\frac{\delta}{\delta\phi_b} - 2g\epsilon_{abc}J_{b\mu}\frac{\delta}{\delta J_c^\mu}\Big)\Delta\Gamma^{(n>0)} = 0$$

can be written

$$s \quad \Big(\Delta\Gamma^{(n>0)}\Big) = 0$$

Phenomenological applications

Once the theory is regularized in a symmetric way, the physical interpretation is far from been settled. We see two possible attitudes. One can state that the theory is given by minimal subtraction (a two parameter theory) or one can use the whole freedom and accept all possible finite renormalizations. In the second case the theory is still predictive.

Limits from the linear sigma model

$$S = \frac{1}{2}\int d^Dx\,(\partial^\mu\phi_a\partial_\mu\phi_a + \partial^\mu\phi_0\partial_\mu\phi_0) - \frac{\lambda^2}{4}\int d^Dx\left(\phi_a\phi_a + \phi_0\phi_0 - v^2\right)^2$$

By using Smirnov's technique one can consider both limits

- Small momenta in comparison with $m_\sigma = \sqrt{2}\lambda v$ with λ fixed. At one loop the leading terms yield a nonlinear sigma model.

- Infinite $m_\sigma = \sqrt{2}\lambda v$ by keeping v fixed. At one loop the limit is singular. The $\ln m_\sigma$ terms are removed by a fine tuning one the nonlinear sigma model.

- Minimal subtraction is used both in the linear and nonlinear case.

- The correspondence is through the $\vec{\phi}$–1PI irreducible vertex functional

$$\tilde{\Gamma}_{\mathrm{LIN}}(\vec{\phi}, J, K_0) \equiv W_{\mathrm{LIN}}(\vec{K}, K_0, J) - \int d^D x \vec{\phi} \vec{K}$$

- with the condition

$$\frac{\delta}{\delta K_0} \tilde{\Gamma}_{\mathrm{LIN}} \Big|_{\vec{\phi}, J, K_0 = 0} = \frac{\delta}{\delta K_0} \Gamma_{\mathrm{NL}} \Big|_{\vec{\phi}, J, K_0 = 0} .$$

- Finite tuning is used on the nonlinear sigma model in order to match the residuum of the $\vec{\phi}$ fields.