

Predictions and Recent Results in SUSY on the Lattice

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Recent reviews

1. S. Catterall, [hep-lat/0509136](#), (Plenary talk at Lattice 2005), “Dirac-Kahler fermions and exact lattice supersymmetry”.
2. D. B. Kaplan, [hep-lat/0309099](#), (Plenary talk at Lattice 2003), “Recent developments in lattice supersymmetry”.
3. A. F., [hep-lat/0210015](#), (Plenary talk at Lattice 2002), “Supersymmetry on the lattice”
4. I. Montvay, [hep-lat/0112007](#), (Review)
5. I. Montvay, [hep-lat/9709080](#), (Plenary talk at Lattice 1997), “SUSY on the lattice”

Motivation

Understanding supersymmetric theories is a challenging and fascinating problem

They are expected to be relevant in the next future experiments

Strongly interacting supersymmetric gauge theories are much studied since in many respects they resemble Quantum Chromodynamics

While much is known analytically, the hope is that a lattice formulation of supersymmetric gauge theories would provide non-perturbative dynamics and additional information for supersymmetry.

SYM dynamics: open questions

The basic feature of SYM dynamics (similar to QCD):

- Confinement
- Spontaneous chiral symmetry breaking

$$U(1)_\lambda \xrightarrow{\text{anomaly}} Z_{2N_c} \xrightarrow{\text{spontaneous}} Z_2$$

Glino condensation: $\langle \lambda\lambda \rangle \neq 0$

- Low-lying mass spectrum
- SUSY Ward-Takahashi identity (WTi) (anomaly?)

This are **non-perturbative effects**

\implies Lattice formulation

Outline

- Overview of the physics of the $N = 1$ SYM theory
- Lattice formulation of $N = 1$ SYM theory
 - Curci-Veneziano: [Wilson fermions](#)
 - Kaplan-Schmaltz: [Domain wall fermions](#)
- Numerical simulations
 - Chiral symmetry breaking
 - Low-lying mass spectrum
 - SUSY WTi → Renormalization constants for the supercurrent.
 - One-loop lattice perturbation theory.

- Exact supersymmetry on the lattice?
 - $N = 1$ Wess-Zumino
 - $N = 2$ SYM theory in $d = 2$
- Conclusions and perspectives

The Model

The continuum action of $N = 1$ SYM and gauge group $SU(N_c)$ reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2}\bar{\lambda}^a \gamma_\mu (\mathcal{D}_\mu \lambda)^a + m_{\tilde{g}} \bar{\lambda}^a \lambda^a,$$

$\lambda = \lambda^a T^a$ is a Majorana spinor in the adjoint representation of the gauge group that satisfies the Majorana condition

$$\bar{\lambda} = \lambda^T C, \quad \lambda = C \bar{\lambda}^T$$

The gluon fields are represented by

$$\begin{aligned} A_\mu &= -ig A_\mu^a T^a \\ F_{\mu\nu} &= -ig F_{\mu\nu}^a T^a \\ \mathcal{D}_\mu \lambda^a &= \partial_\mu \lambda^a + gf_{abc} A_\mu^b \lambda^c. \end{aligned}$$

The action has for $m_{\tilde{g}} = 0$ a supersymmetry respect to the SUSY transformations.

The continuum SUSY transformations read

$$\begin{aligned}\delta A_\mu(x) &= -2g\bar{\lambda}(x)\gamma_\mu\varepsilon \\ \delta\lambda(x) &= -\frac{i}{g}\sigma_{\rho\tau}F_{\rho\tau}(x)\varepsilon \\ \delta\bar{\lambda}(x) &= \frac{i}{g}\bar{\varepsilon}\sigma_{\rho\tau}F_{\rho\tau}(x)\end{aligned}$$

where $\sigma_{\rho\tau} = \frac{i}{2}[\gamma_\rho, \gamma_\tau]$ and ε is a global Grassmann parameter with Majorana properties.

These transformations relate fermions and bosons.

They leave the action invariant and commute with the gauge transformations so that the resulting Noether current $S_\mu(x)$ is gauge invariant.

For $N = 1$ SYM theory the supercurrent is

$$S_\mu = -F_{\rho\tau}^a\sigma_{\rho\tau}\gamma_\mu\lambda^a.$$

Classically the Noether theorem is conserved

$$\partial_\mu S_\mu = 0,$$

(if the fields satisfy the eq. of motion). Furthermore, it fulfills a spin $3/2$ constraint

$$\gamma_\mu S_\mu = 0.$$

SUSY WTi

The existence of the renormalized supercurrent S_μ^R is assumed

$$\partial_\mu S_\mu^R = 2m_R \chi_R$$

where

$$\chi_R = Z_\chi \chi, \quad \chi \equiv \frac{1}{2} F_{\mu\nu}^a \sigma_{\mu\nu} \lambda^a.$$

m_R is the renormalized gluino mass.

- SUSY occurs for $m_R = 0$.
- The non-vanishing of m_R describes a **soft breaking** of SUSY.

Chiral symmetry breaking

Introducing a non zero gluino mass term

$$\mathcal{L}_{mass} = m_{\tilde{g}} \bar{\lambda}^a \lambda^a$$

breaks SUSY softly \rightarrow Non-renormalization theorem and cancellation of divergencies are preserved

Girardello & Grisaru '82

In the massless case, the global chiral symmetry is $U(1)_\lambda$

$$\lambda \rightarrow e^{-i\varphi\gamma_5} \lambda, \quad \bar{\lambda} \rightarrow \bar{\lambda} e^{-i\varphi\gamma_5}.$$

It is anomalous ($J_\mu^5 = \bar{\lambda} \gamma_\mu \gamma_5 \lambda$)

$$\partial_\mu J_\mu^5 = \frac{N_c g^2}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a.$$

The anomaly leaves a Z_{2N_c} subgroup of $U(1)_\lambda$ unbroken

These transformations are equivalent to

$$m_{\tilde{g}} \rightarrow m_{\tilde{g}} e^{-2i\varphi\gamma_5}, \quad \Theta_{SYM} \rightarrow \Theta_{SYM} - 2N_c\varphi$$

In the SUSY case $m_{\tilde{g}} = 0$, $U(1)_\lambda$ symmetry is unbroken if

$$\varphi = \varphi_k \equiv \frac{k\pi}{N_c}, \quad (k = 0, 1, \dots, 2N_c - 1)$$

Z_{2N_c} is expected to be spontaneously broken to Z_2 by $\langle \lambda\lambda \rangle \neq 0$

Witten '82

Consequence of this spontaneous chiral symmetry breaking is

\implies First order phase transition at $m_{\tilde{g}} = 0$

\implies **Existence of N_c degenerate ground states with different orientations of the gluino condensate**

$$\langle \lambda\lambda \rangle = c\Lambda^3 e^{\frac{2\pi ik}{N_c}} \quad (k = 0, \dots, N_c - 1)$$

Dependence of the gauge group

- **case $SU(2)$:** Two degenerate ground states with opposite signs of the gluino condensate, $\langle\lambda\lambda\rangle < 0$, $\langle\lambda\lambda\rangle > 0$
- **case $SU(3)$:** There are three degenerate vacua at $k = k_c$

Magnitude of the gluino condensate

The value of the gluino condensate in $N = 1$ SYM theory has been calculated using two different methods (which gives different results) the $\frac{4}{5}$ puzzle !

- Based on weak-coupling instanton (WCI) calculations

$$\langle \lambda\lambda \rangle = c\Lambda^3$$

$$\Lambda = M_{PV} \left(\frac{16\pi^2}{3N_c g^2} \right)^{1/3} \exp\left(-\frac{8\pi^2}{3N_c g^2} \right)$$

with $c = 6$.

Affleck, Dine & Seiberg '84
Novikov, Shifman, Vainshtein & Zakharov '85
Shifman & Vainshtein '88

- Based on strong-coupling instanton (SCI) calculations with $c = \frac{4}{5}$

Novikov, Shifman, Vainshtein & Zakharov '83
Rossi & Veneziano '84
Amati, Rossi & Veneziano '84

Discussions about the two methods

Amati, Konishi, Meurice, Rossi & Veneziano '88
Kovner & Shifman '97
Hollowood, Khoze, Lee & Mattis '00
Ritz & Vainshtein '00

More recently, a third elegant method has been utilized in which the gluino condensate is directly calculated in the semiclassical approx.

Davies, Hollowood, Khoze & Mattis '99
Davies, Hollowood & Khoze '00

This method gives results in agreement with the weak-coupling calculations (Affleck et al.) and

Finnell & Pouliot '95

Still another method makes use of exact solution solutions of $N = 2$ SYM theories

Seiberg & Witten '94
Argyres & Faraggi '95

One first compute the gluino condensate in the $N = 2$ SYM perturbed by the adjoint mass term and then decouples the adjoint matter fields, going back to pure $N = 1$ SYM. Results are equivalent to the weak-coupling instanton calculation.

Also has been applied to softly $N = 2$ SYM

Konishi & Ricco '03

confirm the correctness of the weak-coupling instanton calculation.

Light hadron spectrum

Veneziano and Yankielowicz '82 proposed an effective action to study the low energy behavior of the SYM theory.

To construct the action they identify all degrees of freedom they expect to govern the low energy dynamics. These are gauge invariant and colorless composite fields

$$\begin{aligned} &F_{\mu\nu}^a F_{\mu\nu}^a \\ &F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a \\ &\bar{\lambda}^a \lambda^a \\ &\sigma_{\mu\nu} F_{\mu\nu}^a \lambda^a \end{aligned}$$

The first three operators are known in QCD while $\chi = \sigma_{\mu\nu} F_{\mu\nu}^a \lambda^a$ is a new type of composite operator formed by the gluino λ and the gauge field F .

The fields can be combined to form the chiral superfield

$$S(x, \theta) = \phi(x) + \sqrt{2}\theta\chi(x) + \theta\theta F(x)$$

$$\phi = \frac{\beta(g)}{2g}(\psi_w)^\alpha(\psi_w)_\alpha, \quad \sqrt{2}\chi_\alpha = -\frac{\beta(g)}{2g}(-i(\psi_w)_\alpha D + (\sigma^{\mu\nu}\psi_w)_\alpha F_{\mu\nu}),$$

$$F = -\frac{\beta(g)}{g} \left\{ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{i}{2}\psi_w\sigma^\mu\partial_\mu\bar{\psi}_w - \frac{i}{8}F_{\mu\nu}\varepsilon_{\mu\nu\rho\tau}F_{\rho\tau} + \frac{i}{2}\partial_\mu J_\mu^5 + \frac{1}{2}D^2 \right\}.$$

The effective \mathbf{VY} action is

$$\mathcal{L}_{eff} = \frac{1}{\alpha}(S^\dagger S)^{1/3}|_D + \gamma[(S \log \frac{S}{\mu^3} - S)|_F + h.c.].$$

Expanding the **effective action** around its minimum, it is found the low-lying spectrum forming a supermultiplet of the Wess-Zumino type consist of

- A scalar meson $\phi = \bar{\lambda}^a \lambda^a$. In analogy with QCD. (The gluino is in the adjoint representation). We call this particle $a - f_0$.
- A massive Majorana fermion $\chi = \sigma_{\mu\nu} F_{\mu\nu}^a \lambda^a$ called gluino-gluon.
- A pseudoscalar meson $\phi_p = \bar{\lambda}^a \gamma_5 \lambda^a$, called the $a - \eta'$.

It is not clear why **glueballs** should be absent in the low-lying spectrum.

The introduction of a non-zero gluino mass breaks susy softly and leads to a splitting of the multiplet.

How the spectrum of glueballs, gluinoballs and gluino-glueballs are influenced by the soft SUSY breaking due to a non-zero gluino mass $m_{\tilde{g}} \neq 0$

$$\begin{aligned}M_{a-\eta'} &= N_c \alpha \Lambda + \frac{40\pi^2 |m_{\tilde{g}}|}{3N_c} + \dots \\M_{a-\chi} &= N_c \alpha \Lambda + \frac{48\pi^2 |m_{\tilde{g}}|}{3N_c} + \dots \\M_{a-f_0} &= N_c \alpha \Lambda + \frac{56\pi^2 |m_{\tilde{g}}|}{3N_c} + \dots\end{aligned}$$

Evans, Hsu & Schwetz '97

The range of applicability of the linear mass formulae is not known because the unknown magnitude of the constants and of the higher order terms.

The question how to include glueballs in the low energy scenario has been addressed by

Farrar, Gabadadze & Schwetz '98

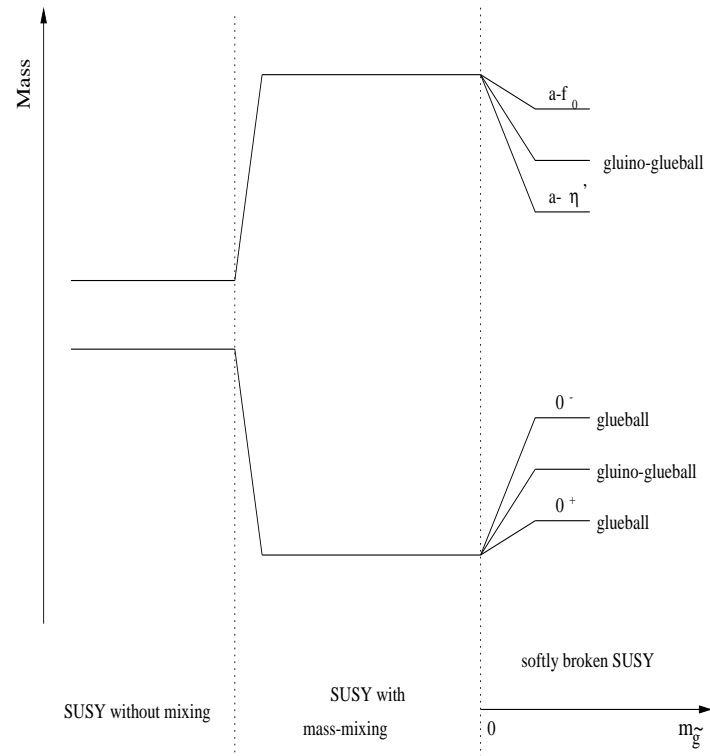
They introduce an extra term in the effective action that gives the dynamics for the glueballs.

For unbroken supersymmetry the masses of these two supermultiplets are **not identical**. The heavier supermultiplet corresponds to the **VY** multiplet. **The lighter one contains**

- A 0^- glueball $\approx F_{\mu\nu}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.
- A gluino-gluon ground state.
- A 0^+ glueball $\approx F_{\mu\nu}F_{\mu\nu}$.

In the low effective action of **FGS** there is a possible non-zero mixing between the states in the two light supermultiplets. In particular there can be a mixing of the $a - f_0$ gluinoball and 0^+ glueball.

Masses



Lattice formulation of SYM theory

The question of whether it is possible to formulate SUSY theories on the lattice has been addressed in the past by several authors

- The lattice regularized theory is not SUSY as the Poincaré invariance (a sector of the superalgebra) is lost.

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\sigma_{\alpha\beta}^\mu P_\mu$$

Not a severe problem. Calculating at several lattice spacing a and then take the limit $a \rightarrow 0$. No fine tuning is needed.

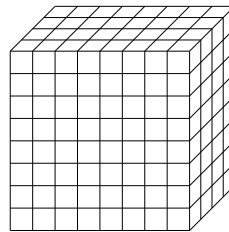
- If there are scalar mass terms in the SUSY theory that break SUSY. Since this operators are relevant fine tuning is necessary to cancel their contributions.
- A naive regularization of fermions results in the doubling problem

Nielsen & Ninomiya '81

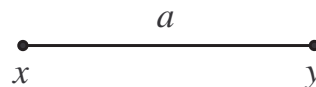
- wrong number of fermions and violation of the balance between bosons and fermions
- The problem can be treated as in QCD. This is the case of $N = 1$ SYM.

Lattice Gauge theory: General Remarks

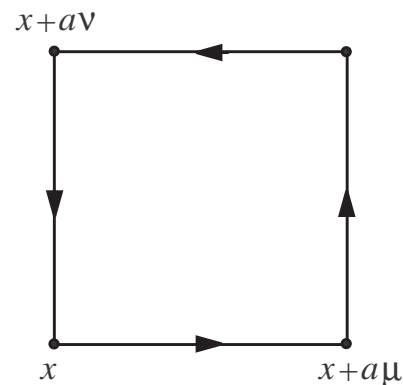
- Discretization of space-time is achieved by introducing a euclidean space-time lattice with spacing a and a volume $L^3 \cdot T$. The inverse lattice spacing a^{-1} acts as an UV cutoff.



- The quark and antiquark fields $\psi(x), \bar{\psi}(x)$ live in the lattice sites x .
- Gauge fields are represented by the link variable $U_\mu(x)$ which are group elements $\in SU(N)$ associated with straight-line path connecting nearest neighbour pairs of lattices sites.



- Gauge invariant expressions on the lattice are traces of products of link variables along a closed path. The most elementary one is the *Plaquette variable*



$$P_{\mu\nu}(x) = U_{\mu}(x)U_{\nu}(x + a\hat{\mu})U_{\mu}^{\dagger}(x + a\hat{\nu})U_{\nu}^{\dagger}(x)$$

- And can be used to construct the *lattice Yang-Mills action* that should reduce in the continuum limit to the classical value. *Wilson propose the simplest one*

$$S_W = \sum_P S_P = \frac{1}{2} \beta \sum_x \sum_{\mu\nu} \left(1 - \frac{1}{2N} \text{Tr}(P_{\mu\nu}(x) + P_{\mu\nu}^\dagger(x)) \right)$$

- Introducing the gauge field variables by

$$U_\mu(x) \equiv \exp ig_0 a A_\mu^b(x) T^b$$

- and using Baker-Campbell-Hausdorff

$$\begin{aligned} P_{\mu\nu} &\simeq e^{ig_0 a^2 F_{\mu\nu}(x)} \\ &\simeq 1 + ig_0 a F_{\mu\nu}(x) - \frac{g_0^2 a^2}{2} F_{\mu\nu}(x) F_{\mu\nu}(x) \end{aligned}$$

in the limit $a \rightarrow 0$

$$S_W = \sum_x \sum_{\mu\nu} \left(a^4 \frac{\beta g_0^2}{2N} \text{Tr} F_{\mu\nu} F_{\mu\nu} + O(a^6) \right).$$

- So the continuum limit is

$$S_W = \int d^4x \frac{1}{2} \text{Tr} F_{\mu\nu} F_{\mu\nu},$$

if we define β to be

$$\beta = \frac{2N}{g_0^2}.$$

- The expectation value of an observable Ω that depends on the variable U

$$\langle \Omega \rangle = \frac{1}{Z} \int \prod_{x,\mu} dU_\mu(x) \Omega(U) e^{-S_w(U)},$$

and the functional integral Z is determined by requiring $\langle 1 \rangle = 1$.

- An efficient way to computing $\langle \Omega \rangle$ would consist in generating a sequence of link variable configurations with a probability distribution given by the Boltzmann factor $e^{-S(C)}$, where $S(C)$ is the action related to this configuration.
- Monte Carlo method

$$\langle \Omega_L \rangle (\beta) \cong \frac{1}{n} \sum_{i=1}^n \Omega_L(C_i)$$

with $\{C\}_i$, ($i = 1, \dots, n$) denote the link configurations generated.

- The Monte Carlo method consists in producing a sequence of configurations $U^{(1)} \rightarrow U^{(2)} \rightarrow U^{(3)} \rightarrow \dots$ with the appropriate probabilities in a statistical way. This is done *by the computer*.

Fermions on the Lattice

- Recently, following the rediscovery of the Ginsparg-Wilson relation (1982), it has emerged that chiral gauge theories can be put on the lattice in a consistent way:
 - The overlap (Narayanan-Neuberger 1993,1995,1998)
 - Domain wall fermions (Kaplan-Shamir 1992, 1993, 1994)
 - Perfect action (Hasenfratz-Niedermayer 1994, 1998).

This was believed to be impossible for a long time (Nielsen-Ninomiya, 1981, no-go theorem).

- A naive formulation of fermions on the lattice fails.

$$S_F = \frac{1}{2} \sum_x \sum_\mu \bar{\psi}(x) (\gamma_\mu \Delta_\mu + m) \psi(x) + h.c.$$

and the resulting propagator is

$$\tilde{\Delta}(k) = \frac{-i \sum_\mu \gamma_\mu \sin k_\mu + m}{\sum_\mu \sin^2 k_\mu + m^2}$$

- There is a pole for small k_μ representing the physical particle, but additional poles near $k_\mu = \pm\pi$ appears. S_F describes 16 instead of 1 particle.
→ Doubling problem.
- Two popular choices introduced in order to deal with this problem:
 - Wilson fermions: Get rid of the doubling species but breaks chiral symmetry explicitly by the Wilson term.
 - Staggered fermions (Kogut-Susskind): Reduce from 16 to 4 fermions and for massless fermions a chiral $U(1) \oplus U(1)$ symmetry remains.
- In the Wilson formulation the bare mass m is hidden in the hopping parameter by the relation $k = \frac{1}{8r+2m_0}$.

- Take Wilson's or staggered fermions for the quarks fields $\psi_{\alpha c}^f(x)$, the complete action is $S = S_W + S_F$. And for an observable we write down

$$\langle \Omega \rangle = \frac{1}{Z} \int \prod_{x,\mu} dU_\mu(x) \int \prod_x d\bar{\psi}(x) d\psi(x) \Omega e^{-S_W - S_F},$$

- After integrating out the quarks fields the expectation value reads

$$\langle \Omega \rangle = \frac{1}{Z} \int \prod_{x,\mu} dU_\mu(x) \prod_f \det(D + m_f) \Omega e^{-S_W},$$

where D is the Dirac operator.

Two ways to study SUSY on the lattice

- Construct non-SUSY lattice theories with a SUSY continuum limit.
 - N=1 Super-Yang Mills
(with Wilson fermions – Curci and Veneziano formulation)

- Keep some exact algebra of SUSY on the lattice in order to recover the continuum limit with no (or less) fine tuning of parameters of the action.
 - N=1 Super-Yang Mills
(with Domain Wall-fermions – Kaplan formulation) → the zero gluino mass term is achieved without fine tuning

Wilson fermions

Propose to give up manifest SUSY on the lattice and restore it in the continuum limit.

Curci & Veneziano '87

SUSY is broken by the lattice, by the Wilson term and a soft breaking due to the gluino mass is present.

- SUSY is recovered in the continuum limit by tuning the bare parameters g and gluino mass $m_{\tilde{g}}$ to the SUSY point.
- The chiral and SUSY limit can be recovered simultaneously at $m_{\tilde{g}} = 0$.

The inclusion of a gluino mass breaks supersymmetry softly and the bare gluino mass has to be tuned numerically to its critical value which is the chiral – SUSY limit – $m_{\tilde{g}} = 0$

Wilson fermions

The Curci and Veneziano action reads

$$S = S_G + S_F,$$

$$S_G = \frac{\beta}{2} \sum_x \sum_{\mu\nu} \left(1 - \frac{1}{N_c} \text{Re Tr } U_{\mu\nu}(x) \right),$$

and $\beta \equiv 2N_c/g_0^2$ correspond to the bare gauge coupling.

$$\begin{aligned} S_F = \text{Tr} & \left\{ \frac{1}{2a} \left(\bar{\lambda}(x) (\gamma_\mu - r) U_\mu^\dagger(x) \lambda(x + a\hat{\mu}) U_\mu(x) \right. \right. \\ & \left. \left. - \bar{\lambda}(x + a\hat{\mu}) (\gamma_\mu + r) U_\mu(x) \lambda(x) U_\mu^\dagger(x) \right) \right. \\ & \left. + \left(m_0 + \frac{4r}{a} \right) \bar{\lambda}(x) \lambda(x) \right\}. \end{aligned}$$

The Grassmann variables λ and $\bar{\lambda}$ are not independent

$$\bar{\lambda} = \lambda^T C, \quad \lambda = C \bar{\lambda}^T.$$

Monte Carlo simulations: A different parametrization is used. The hopping parameter k is

$$k = \frac{1}{2(4 + m_0 a)}$$

The lattice Wilson fermion action

$$S_F \equiv \frac{1}{2} \bar{\lambda} Q \lambda \equiv \frac{1}{2} \sum_x \left\{ \bar{\lambda}_x^a \lambda_x^a - k \sum_{\mu=1}^4 \left[\bar{\lambda}_{x+\hat{\mu}}^a V_{ab,x\mu} (1 + \gamma_\mu) \lambda_x^b + \bar{\lambda}_x^a V_{ab,x\mu}^T (1 - \gamma_\mu) \lambda_{x+\hat{\mu}}^b \right] \right\}$$

Montvay '98

with the adjoint link $V_{ab,x\mu}(x)$ in the adjoint representation

$$\begin{aligned} V_{ab,x\mu} &\equiv V_{ab,x\mu}[U] \equiv \\ &\equiv 2\text{Tr}(U_{x\mu}^\dagger T_a U_{x\mu} T_b) = V_{ab,x\mu}^* = V_{ab,x\mu}^{-1T}. \end{aligned}$$

The path integral over the Majorana fermions gives the **Pfaffian**

$$\int [d\lambda] e^{-\frac{1}{2}\bar{\lambda}Q\lambda} = \int [d\lambda] e^{-\frac{1}{2}\lambda^T C Q \lambda} = Pf(M) = \underbrace{\pm}_{\text{sign}} \sqrt{\det Q}.$$

where $M \equiv CQ = -M^T$ is an antisymmetric matrix.

Algorithm for numerical simulations

The effective action is

$$S_{CV} = \beta \sum_{pl} \left(1 - \frac{1}{2} \text{Tr} U_{pl} \right) - \frac{1}{2} \log \det Q[U].$$

Curci & Veneziano '87

The omitted sign of the Pfaffian can be taken into account by the **reweighting formula**

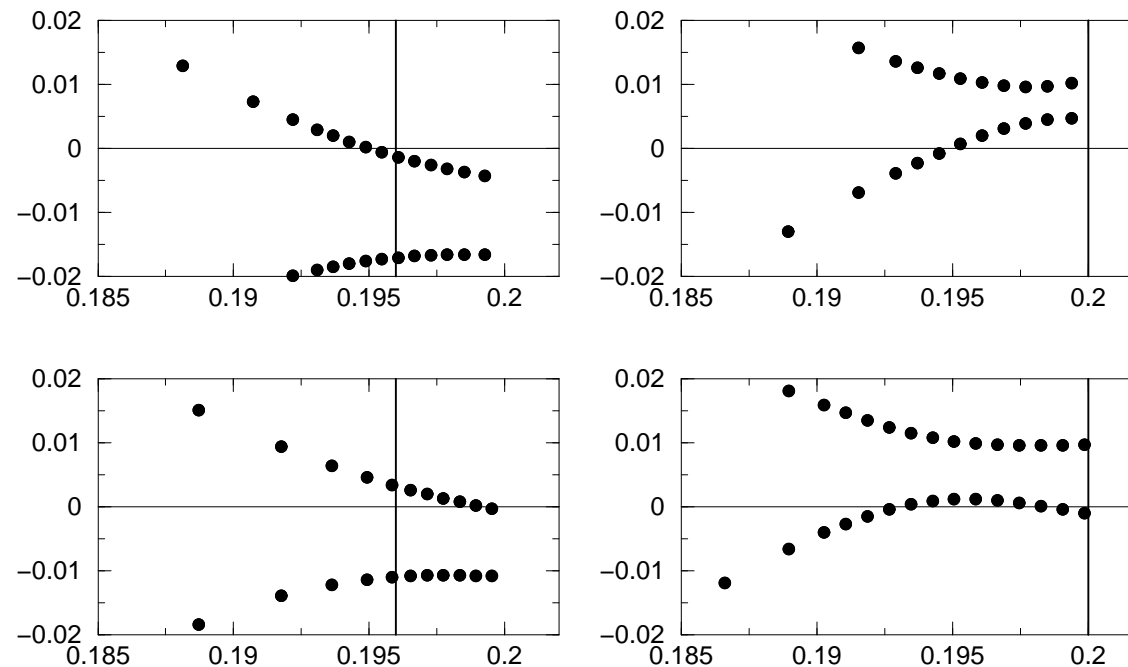
$$\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} \text{sign Pf}(M) \rangle_{CV}}{\langle \text{sign Pf}(M) \rangle_{CV}}$$

may rise to the **sign problem**

Spectral flow method: for the sign.

$$|\text{Pf}(M)| = \prod_{i=1}^{\Omega/2} |\tilde{\lambda}_i|, \quad \implies \text{Pf}(M) = \prod_{i=1}^{\Omega/2} \tilde{\lambda}_i.$$

If the value of an eigenvalue $\tilde{\lambda}_i$ changes sign, the sign of $Pf(M)$ has to change too.



The spectral flow of the hermitian fermion matrix \tilde{Q} for a configuration on $6^3 \times 12$ at $\beta = 2.3$. The value of k in the simulation correspond to the vertical line. Montvay et al. '99. For $k < k_c \rightarrow$ no serious sign problems!

The simulation has been develop in the *Two-step multibosonic algorithm* (TSMB)

Montvay '96,'98

To represent the fermion determinant ones uses a first polynomial $\mathcal{P}_{n_1}^{(1)}(x)$ for a crude approximation realizing a fine correction by another polynomial $\mathcal{P}_{n_2}^{(2)}(x)$

$$\mathcal{P}_{n_1}^{(1)}(x)\mathcal{P}_{n_2}^{(2)}(x) \approx x^{-N_f/2} \quad x \in [\varepsilon, \lambda].$$

The fermion determinant is approximated as

$$\det(Q^\dagger Q)^{N_f} \simeq \frac{1}{\det P_{n_1}^{(1)}(Q^\dagger Q)\det P_{n_2}^{(2)}(Q^\dagger Q)}.$$

Domain wall fermions

A new lattice fermion regulator. **Very nice innovation.** Application of DWF in SUSY theories

Neuberger '98
Kaplan & Schmaltz '00

Monte Carlo simulation for $N = 1$ $SU(2)$ SYM with DWF

Fleming, Kogut & Vranas '01

DWF were introduced by

Kaplan '92,'93

and further developed in

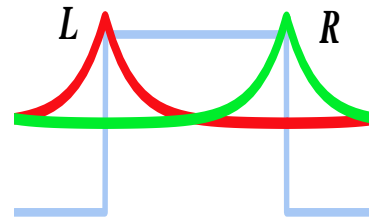
Narayanan & Neuberger '93,'94,'95
Shamir '93
Furman & Shamir '95

Difficulties in using Wilson fermions.

- Need to fine tuning. The Wilson term breaks chiral symmetry
- The Pfaffian. Is not positive definite at finite lattice spacing.

DWF are defined extending space-time to **five dimensions**.

L_s is the size of the fifth dimension.



In the limit $L_s \rightarrow \infty$ chiral symmetry is exact, even at finite lattice spacing.

- There is not need for fine tuning.

The domain wall action is

$$S = S_G(U) + S_F(\Psi, U) + S_{PV}(\Phi, U)$$

$$S_F = - \sum_{x, x', s, s'} \bar{\Psi}_{x, s} (D_F)_{x, s; x', s'} \Psi_{x', s'}$$

The effective action is

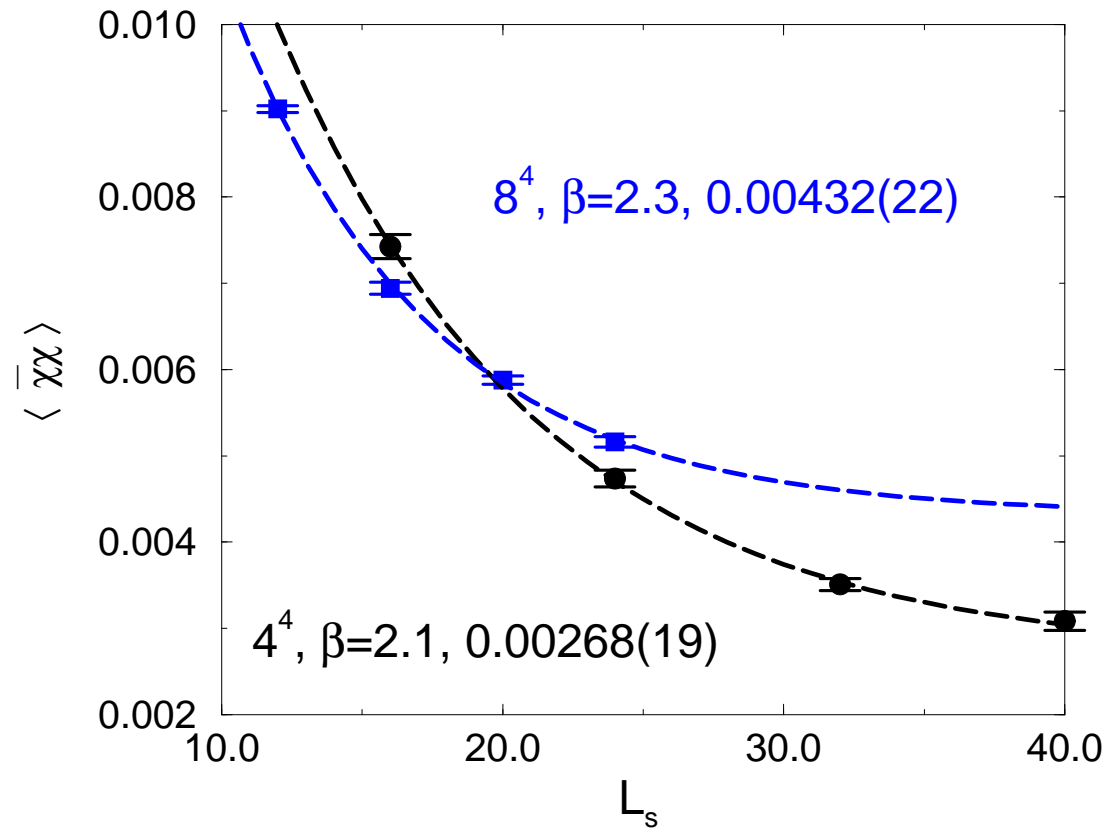
$$S_{KS} = \beta \sum_{pl} \left(1 - \frac{1}{2} \text{Tr} U_{pl} \right) - \frac{1}{2} \log \det D_F[U] \\ + \frac{1}{2} \log \det D_F[m_f = 1; U].$$

Difficulties in using DWF.

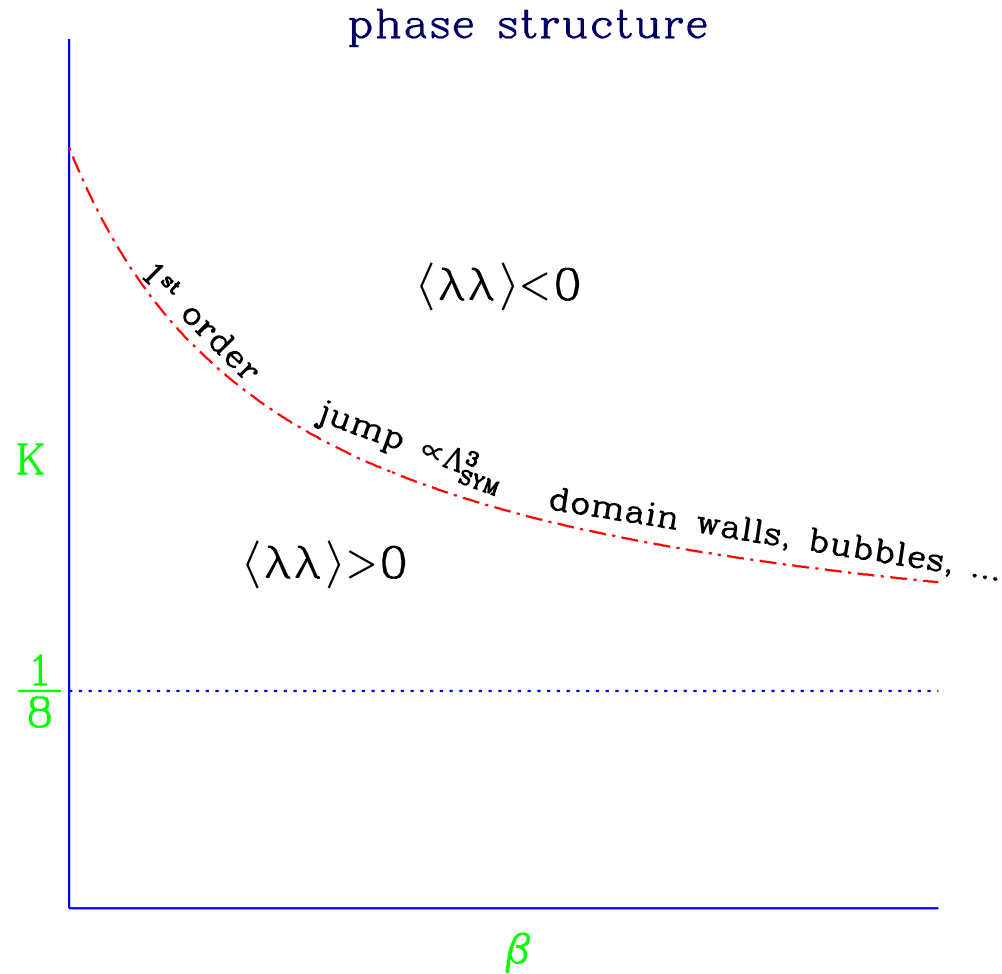
- 2 extra parameters in DWF: L_s and m_0 (m_0 is the domain wall height or five-dimensional mass that controls the number of flavors).

$$m_{eff} = m_0(2 - m_0)[m_f + (1 - m_0)^{L_s}], \quad 0 < m_0 < 2$$

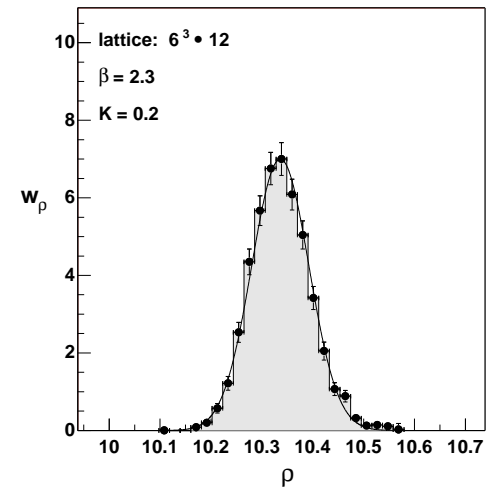
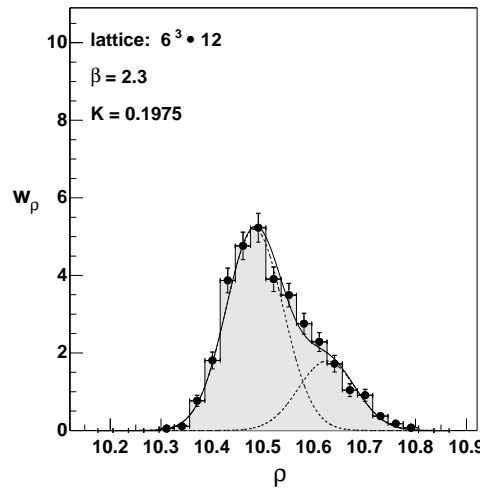
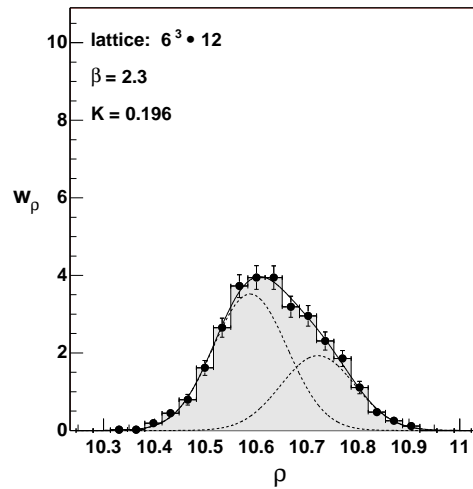
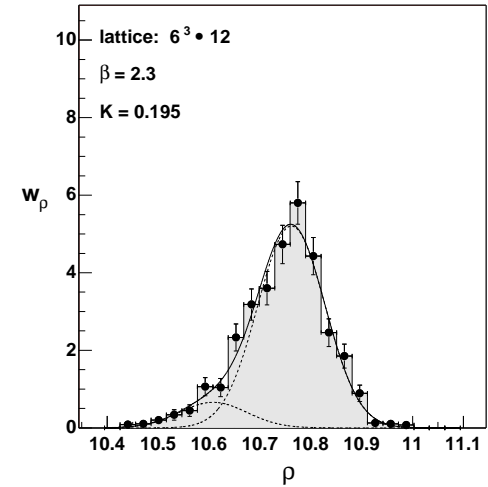
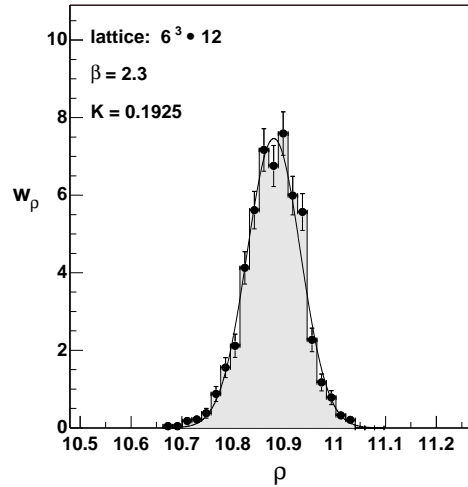
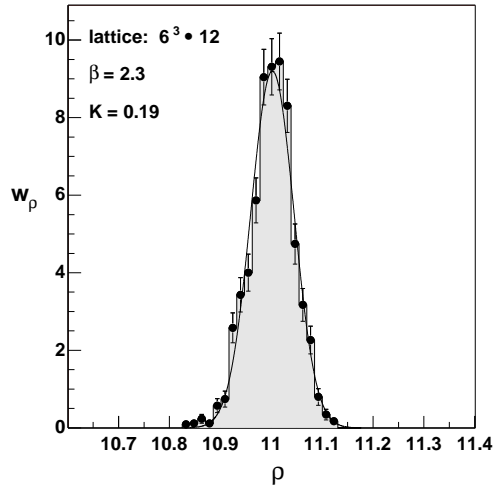
- The two chiralities do not decouple \rightarrow no restoration of chiral symmetry. (Need large values of L_s)
- Harder to simulate than QCD (with Wilson fermions easier to simulate than QCD)



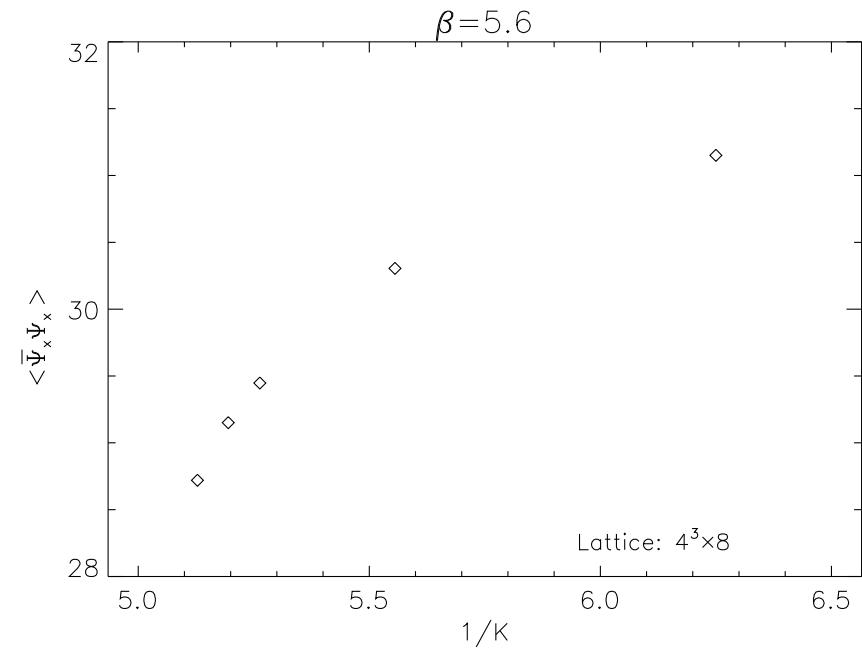
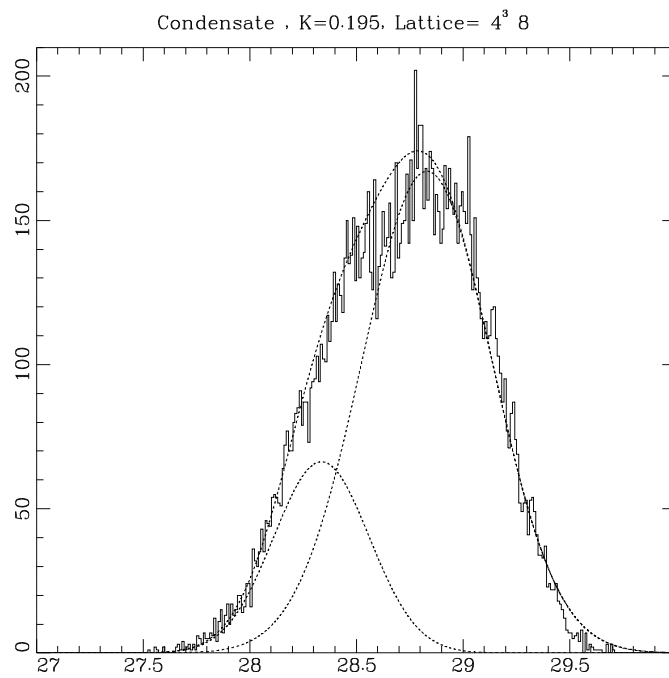
Dynamical gluino condensate at $m_f = 0$ vs L_s on two different lattices. (Fleming, Kogut, Vranas '00)



Expected phase structure of SYM in the (β, k) plane. Dashed line $k = k_c(\beta)$ is a first-order phase transition (or cross-over) at $m_{\tilde{g}} = 0$.

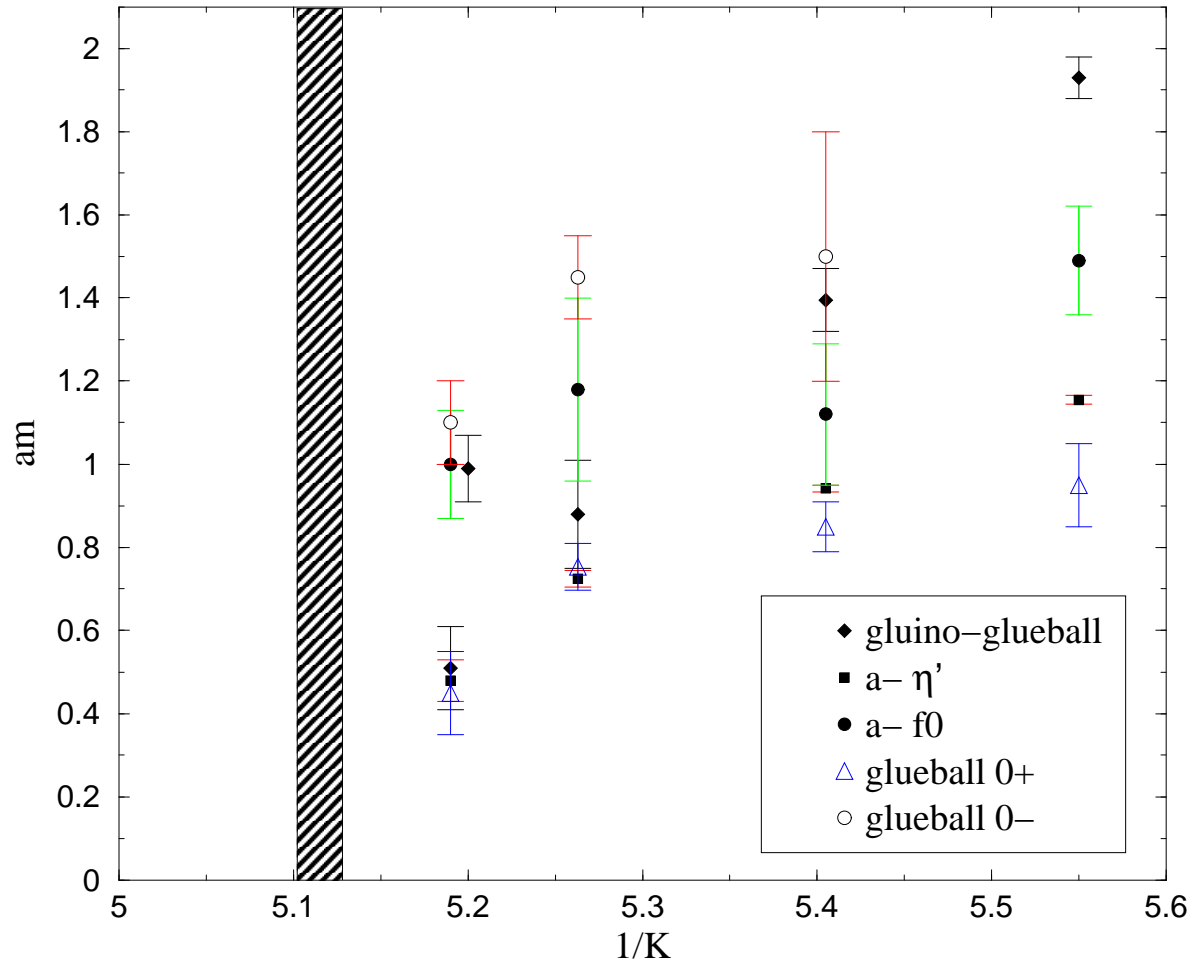


Distribution of the gluino condensate for different k values at $\beta = 2.3$ and lattice size $6^3 \times 12$, $k_c = (0.1955 \pm 0.0005)$ and $N_c = 2$. (Montvay et al., '99)



Distribution of the gluino condensate for $k = 0.195$ at $\beta = 5.6$ and lattice size $4^3 \times 8$ and $N_c = 3$. (Montvay et al., '00)

Light hadron spectrum in lattice units (Montvay et al., '00, '01). $k_c = (0.1955 \pm 0.0005)$.
 Lattice size $12^3 \times 24$.

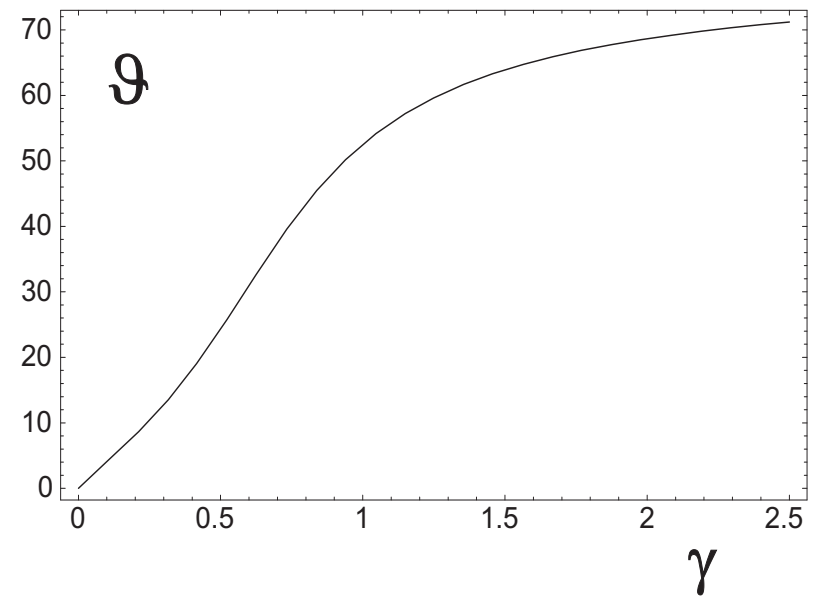
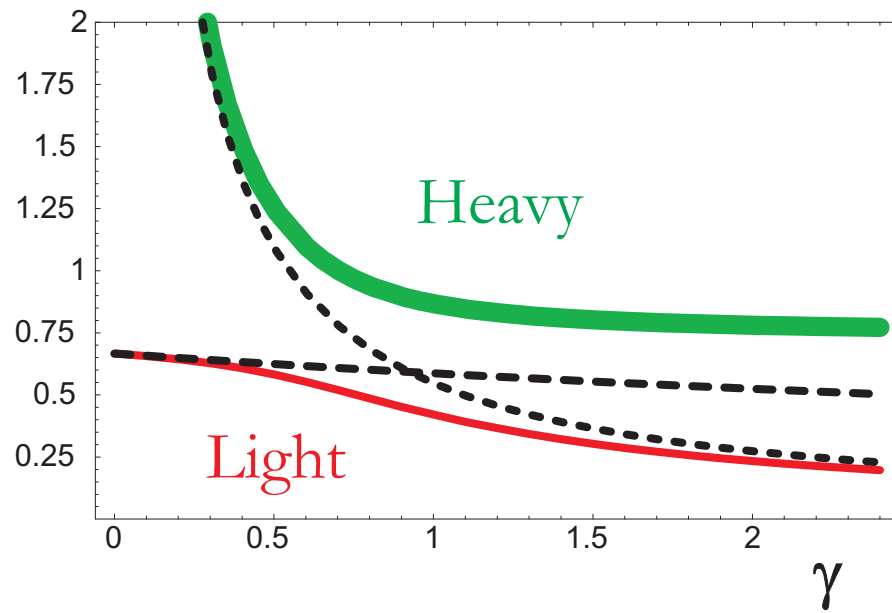


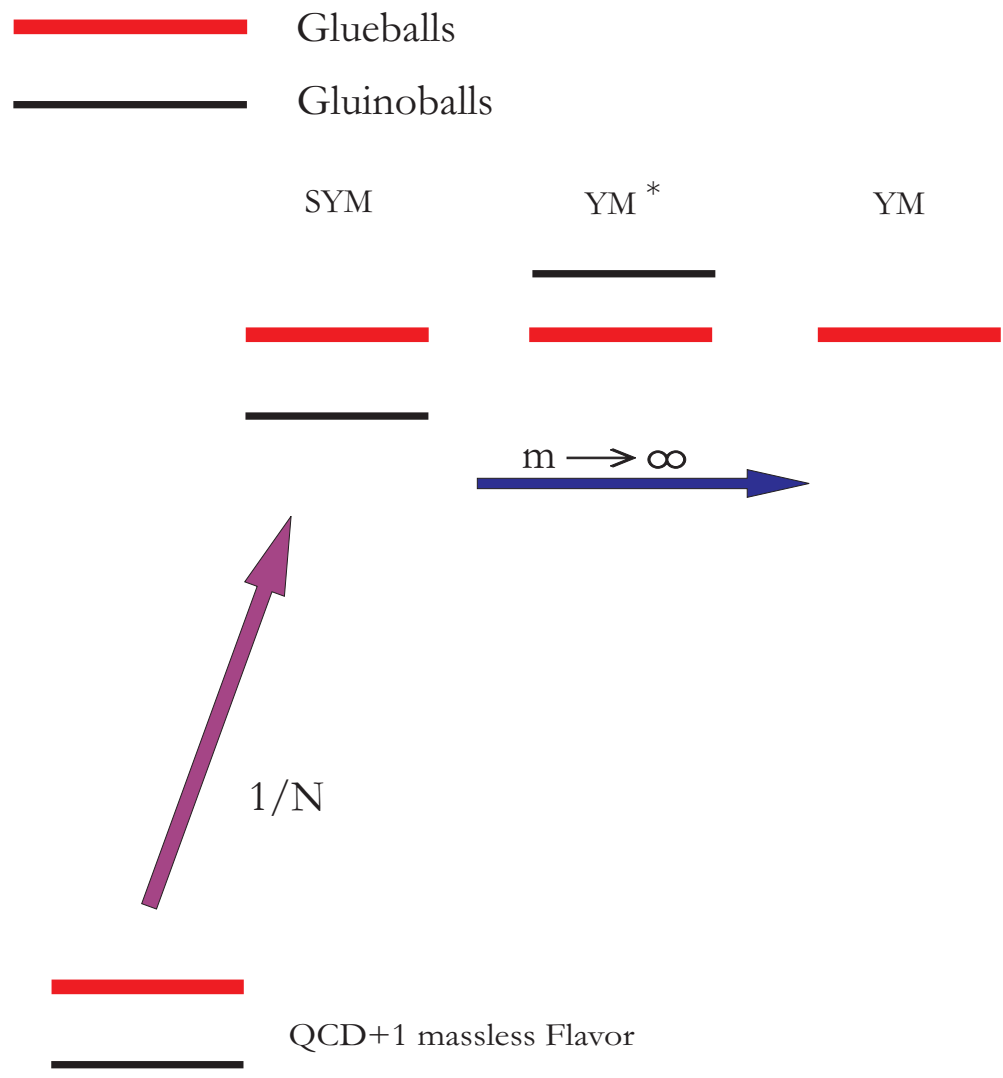
Information on the Super Yang-Mills Spectrum

It can be demonstrated that with an extended Lagrangian approach one can accommodate either the possibility in which the glueballs are heavier or lighter than the gluinoball fields.

If one uses information about ordinary QCD one can deduce that the lightest states in super Yang-Mills are the gluinoballs.

Merlatti and Sannino, hep-th/0404251
Feo, Merlatti and Sannino, hep-th/0408214





Lattice SUSY WTi with Wilson fermions

Another independent way to study the SUSY limit (chiral limit) is defining the SUSY WTi.

In the Wilson formulation, additional breaking terms on the lattice arise from the explicit breaking of supersymmetry.

The SUSY limit is defined to be the point in parameter space in which this breaking terms vanish and the WTi takes *its continuum limit*.

Nevertheless supersymmetry is not fulfilled on the lattice one might still define some SUSY transformations. One choice is

$$\begin{aligned}\delta U_\mu(x) &= -agU_\mu(x)\bar{\xi}\gamma_\mu\lambda(x) - ag\bar{\xi}\gamma_\mu\lambda(x+a\hat{\mu})U_\mu(x), \\ \delta U_\mu^\dagger(x) &= ag\bar{\xi}\gamma_\mu\lambda(x)U_\mu^\dagger(x) + agU_\mu^\dagger(x)\bar{\xi}\gamma_\mu\lambda(x+a\hat{\mu}), \\ \delta\lambda(x) &= -\frac{i}{g}\sigma_{\rho\tau}\mathcal{G}_{\rho\tau}(x)\xi, \\ \delta\bar{\lambda}(x) &= \frac{i}{g}\bar{\xi}\sigma_{\rho\tau}\mathcal{G}_{\rho\tau}(x)\end{aligned}$$

$\mathcal{G}_{\rho\tau}$ is the clover plaquette operator

$$\begin{aligned}\mathcal{G}_{\rho\tau}(x) &= -\frac{1}{8a^2} (U_{\mu\nu}(x) - U_{\nu\mu}(x) + U_{-\mu,-\nu}(x) - U_{-\nu,-\mu}(x) \\ &\quad + U_{\nu,-\mu}(x) - U_{-\mu,\nu}(x) + U_{-\nu,\mu}(x) - U_{\mu,-\nu}(x))\end{aligned}$$

they reduce to the continuum SUSY transformations in the limit $a \rightarrow 0$.

Numerical Simulations

On-shell regime

In a regularization, that breaks SUSY, the bare WTi reads

$$\langle \mathcal{O} \Delta_\mu S_\mu(x) \rangle - 2m_0 \langle \mathcal{O} \chi(x) \rangle + CT = \langle \mathcal{O} X_S(x) \rangle$$

$X_S(x)$ is the term originated by the SUSY breaking of the lattice action.
 $X_S(x)$ is $\simeq a O_{\frac{11}{2}}(x)$

If \mathcal{O} is gauge-invariant $\rightarrow X_S(x)$ is gauge invariant.

In order to renormalize the SUSY WTi on the lattice a possible operator mixing has to be taken into account. X_S mixes with operators of equal or lower dimension (Bochicchio et al. '85).

- $\Delta_\mu S_\mu$
- $\Delta_\mu T_\mu = -\Delta_\mu \mathcal{G}_{\mu\nu}^a \gamma_\nu \lambda^a$
- $\chi = \frac{1}{2} \mathcal{G}_{\mu\nu}^a(x) \sigma_{\mu\nu} \lambda^a(x)$

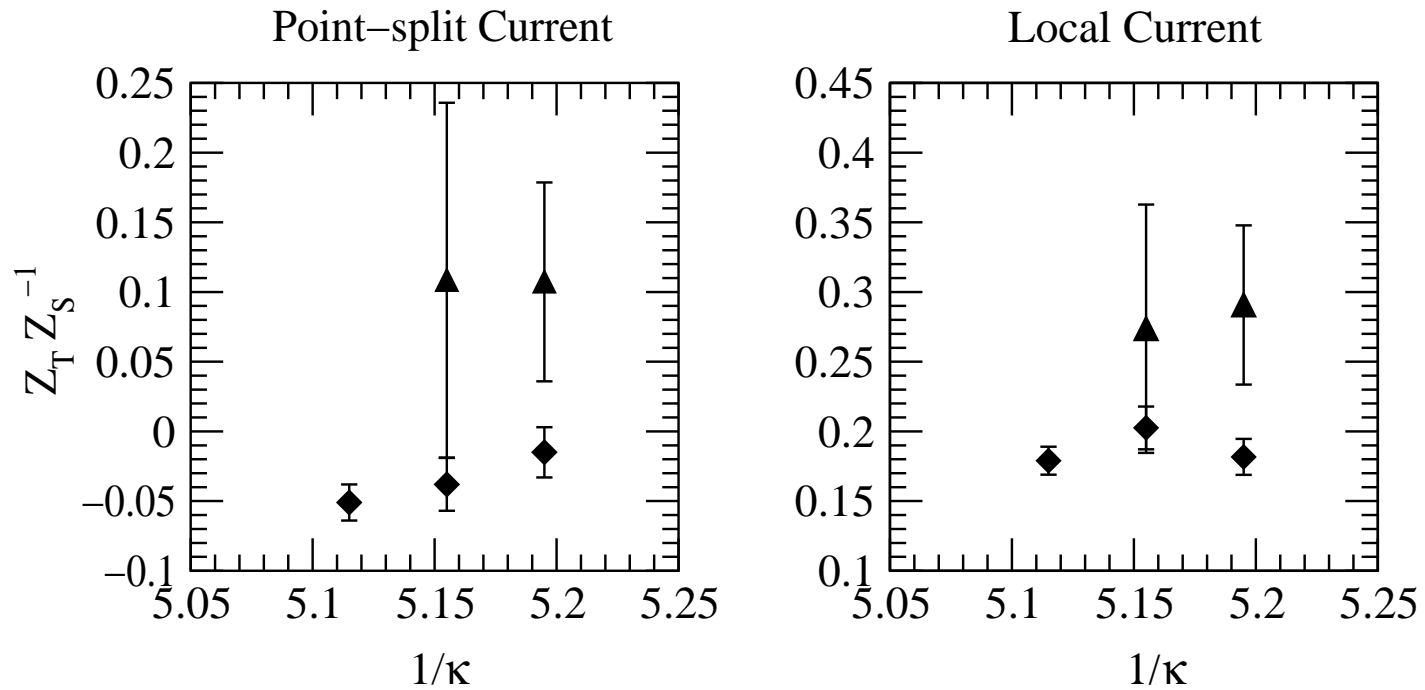
Define a subtracted $\bar{X}_S(x)$, whose expectation value is forced to vanish in the limit $a \rightarrow 0$.

$$\bar{X}_S(x) = X_S(x) + (Z_S - 1)\Delta_\mu S_\mu(x) + 2\tilde{m}\chi(x) + Z_T\Delta_\mu T_\mu(x)$$

The renormalized SUSY WTi

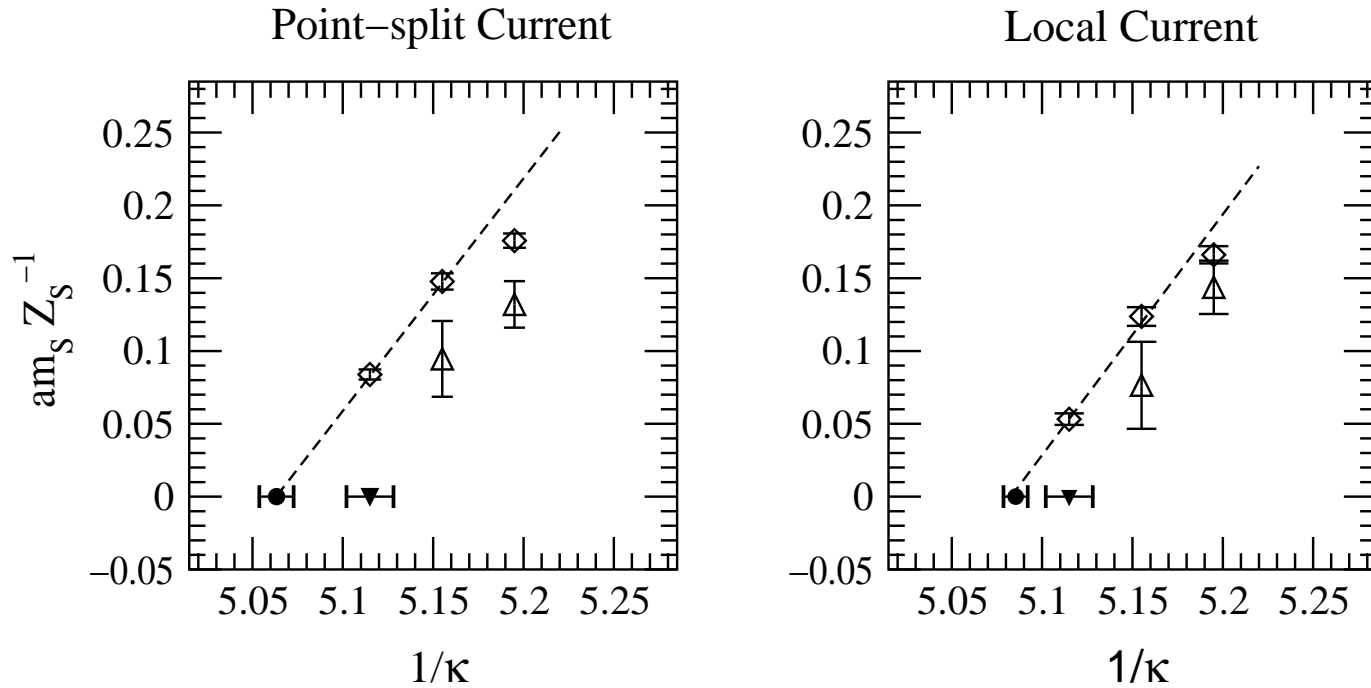
$$\langle \mathcal{O}\Delta_\mu S_\mu(x) \rangle + Z_T/Z_S \langle \mathcal{O}\Delta_\mu T_\mu(x) \rangle = 2m_S/Z_S \langle \mathcal{O}\chi(x) \rangle + O(a)$$

where $m_S = m_0 - \tilde{m}$.



$Z_T Z_S^{-1}$ as a function of $1/k$ with the insertion operator $\chi(x)$ (filled diamonds) and $T_0^{loc}(x)$ (filled triangles). Only for insertion $\chi(x)$, $Z_T Z_S^{-1} = -0.039(7)$ for the point-split current $Z_T Z_S^{-1} = 0.185(7)$ for the local current.

An estimate of $Z_T Z_S^{-1}$ for the point split current at $\beta = 2.3$ at 1-loop perturbative calculation is $Z_T Z_S^{-1} \equiv Z_T|_{1-loop} = -0.074$. (Taniguchi '99). Also $Z_T Z_S^{-1} \equiv Z_T|_{1-loop} = 0.332$ for the local current (Feo, hep-lat/0305020).



$am_s Z_s^{-1}$ as a function of $1/k$ with the insertion operator $\chi(x)$ (diamonds) and $T_0^{loc}(x)$ (triangles). A linear extrapolation is also reported. The filled triangle indicates the determination of the k_c from the first order phase transition $k_c = (0.1955 \pm 0.0005)$.

Here the value of k_c from the SUSY WTi is $k_c = 0.19750(38)$ for the point split current and $k_c = 0.19647(27)$ for the local current.

Exact supersymmetry on the lattice

Four dimensional lattice Wess-Zumino model with GW fermions

We show that it is actually possible to formulate the theory in such a way that the full action is invariant under a lattice supersymmetry transformation at a fixed lattice spacing.

The action and the transformation are written in terms of the Ginsparg-Wilson operator and reduce to their continuum expression in the naive continuum limit $a \rightarrow 0$.

The lattice supersymmetry transformation is non-linear in the scalar fields and depends on the parameters m and g entering in the superpotential.

We also show that the lattice supersymmetry transformation close the algebra, which is a necessary ingredient to guarantee the request of supersymmetry.

Bonini and Feo, hep-lat/0402034, hep-lat/0504010
Feo, hep-lat/0512028 and in preparation.

The Ginsparg-Wilson (GW) relation

$$\gamma_5 D + D \gamma_5 = a D \gamma_5 D$$

implies a continuum symmetry of the fermion action which may be regarded as a lattice form of the chiral symmetry (Lüscher '98).

A solution was given by Neuberger '97,'98

$$D = \frac{1}{a} \left(1 - X \frac{1}{\sqrt{X^\dagger X}} \right), \quad X = 1 - a D_w,$$

where

$$D_w = \frac{1}{2} \gamma_\mu (\nabla_\mu^* + \nabla_\mu) - \frac{a}{2} \nabla_\mu^* \nabla_\mu$$

In terms of the component fields the lattice Wess-Zumino action reads
(Fujikawa 2002) and (Fujikawa and Ishibashi 2002)

$$S_{WZ} = S_0 + S_{int},$$

with

$$S_0 = \sum_x \left\{ \frac{1}{2} \bar{\chi} \left(1 - \frac{a}{2} D_1\right)^{-1} D_2 \chi - \frac{1}{a} (A D_1 A + B D_1 B) \right. \\ \left. + \frac{1}{2} F \left(1 - \frac{a}{2} D_1\right)^{-1} F + \frac{1}{2} G \left(1 - \frac{a}{2} D_1\right)^{-1} G \right\},$$

$$S_{int} = \sum_x \left\{ \frac{1}{2} m \bar{\chi} \chi + m (F A + G B) + \frac{1}{\sqrt{2}} g \bar{\chi} (A + i \gamma_5 B) \chi \right. \\ \left. + \frac{1}{\sqrt{2}} g [F (A^2 - B^2) + 2G (AB)] \right\}.$$

where A , B , F and G are real scalar fields and χ is a Majorana fermion which satisfies the Majorana condition.

The supersymmetric transformations

The total action is invariant under the transformations

$$\delta A = \bar{\varepsilon}\chi = \bar{\chi}\varepsilon$$

$$\delta B = -i\bar{\varepsilon}\gamma_5\chi = -i\bar{\chi}\gamma_5\varepsilon$$

$$\delta\chi = -D_2(A - i\gamma_5 B)\varepsilon - (F - i\gamma_5 G)\varepsilon + gR\varepsilon$$

$$\delta F = \bar{\varepsilon}D_2\chi$$

$$\delta G = i\bar{\varepsilon}D_2\gamma_5\chi,$$

$$R = R^{(1)} + gR^{(2)} + \dots$$

where

$$R^{(1)} = \left(\left(1 - \frac{a}{2}D_1 \right)^{-1} D_2 + m \right)^{-1} \Delta L$$

with

$$\Delta L \equiv \frac{1}{\sqrt{2}} \{ 2(AD_2A - BD_2B) - D_2(A^2 - B^2) + 2i\gamma_5[(AD_2B + BD_2A) - D_2(AB)] \}$$

and for $n \geq 2$

$$R^{(n)} = -\sqrt{2} \left(\left(1 - \frac{a}{2} D_1 \right)^{-1} D_2 + m \right)^{-1} (A + i\gamma_5 B) R^{(n-1)} .$$

$$\left[\left(1 - \frac{a}{2} D_1 \right)^{-1} D_2 + m + \sqrt{2} g (A + i\gamma_5 B) \right] R = \Delta L .$$

It has been shown that the algebra associated to the lattice supersymmetry transformation closes.

The existence of this exact symmetry should be responsible for the restoration of supersymmetry in the continuum limit without fine tuning. We prove this using the WTi to order g , g^2 , g^3 .

Two-point Ward-Takahashi identity

Order g^2

$$\langle \chi_y \bar{\chi}_x \rangle - \langle D_{2yz} (A_z - i\gamma_5 B_z) A_x \rangle - \langle (F_y - i\gamma_5 G_y) A_x \rangle + g \langle R_y A_x \rangle = 0.$$

This WTi is satisfied at fixed lattice spacing and in the continuum limit.

This WTi determines the finite part of the scalar and fermion renormalization wave functions which automatically leads to restoration of susy in the continuum limit. **In particular these wave functions coincide in this limit.**

$N = 2$ $d = 2$ lattice Super Yang-Mills theory

Twisted formulation using Dirac-Kaehler fermions (Catterall, D'Adda et al.), while Sugino uses another formulation.

The idea is that the supercharges are to be interpreted as matrices in the twisted picture,

$$q = QI + Q_\mu \gamma_\mu + Q_{12} \gamma_1 \gamma_2$$

where Q, Q_μ, Q_{12} are the twisted supercharges and satisfy the algebra

$$\{Q, Q\} = \{\tilde{Q}, \tilde{Q}\} = \{Q, \tilde{Q}\} = \{Q_\mu, Q_\nu\} = 0, \quad \{Q, Q_\mu\} = P_\mu, \quad \{\tilde{Q}, Q_\mu\} = -\varepsilon_{12} P_\mu$$

Catterall and Sugino write down the lattice action as a Q-exact form while D'Adda et al. keep all the supercharges exact.

Conclusions & Perspectives

A big effort has been made in order to describe supersymmetry on the lattice.

Traditional Wilson fermions have been used in realistic computations with nice results.

Improved chiral fermions results are starting too.

Exact supersymmetry on the lattice?

- Improving lattice SUSY seems to be a difficult task for gauge theories because on the lattice the gauge field and the fermions are treated in a very different way.
 - Exact formulations for $N = 1$ Wess-Zumino model and $N = 2$ SYM in $d = 2$ have been achieved!
 - $N = 1$ SYM in $d = 4$??