HOLOGRAPHY AND GENERALIZED UNCERTAINTY PRINCIPLE

FABIO SCARDIGLI
CENTRA - Departamento de Fisica, Instituto Superior Tecnico,
Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal
E-mail: scardigli@fisica.ist.utl.pt

ROBERTO CASADIO
Dipartimento di Fisica, Università di Bologna and I.N.F.N., Sezione di Bologna, via Irnerio 46, 40126 Bologna, Italy
E-mail: casadio@bo.infn.it

Generalized uncertainty principle(s) (GUPs) in $4(+n)$ dimensions and their holographic properties have been investigated in Ref. 4. In this short paper we review the fact that the expected holographic scaling holds only for one specific kind of GUP and only in four dimensions: when extra spatial dimensions are admitted, holography seems to be destroyed. This might suggest that holographic principle could constrain space-time to be four dimensional.

1. Introduction

During the last years many efforts have been devoted to clarifying the role played by the existence of extra spatial dimensions in the theory of gravity [1,2]. One of the most interesting predictions drawn from the theory is that there should be measurable deviations from the $1/r^2$ law of Newtonian gravity at short (and perhaps also at large) distances. Such new laws of gravity would imply modifications of those Generalized Uncertainty Principles (GUP's) (see Ref. 5) designed to account for gravitational effects in the measure of positions and energies.

On the other hand, the holographic principle is claimed to apply to all of the gravitational systems. The existence of GUP's satisfying the holography in four dimensions (one of the main examples is due to Ng and Van Dam [3]) led us to explore the holographic properties of the GUP's extended to the brane-world scenarios [4]. The results, at least for the examples we considered, are quite surprising. The expected holographic scaling indeed seems to hold only in four dimensions, and only for the Ng and van Dam's GUP. When extra spatial dimensions are admitted, the holography is destroyed. This fact allows two different interpretations: either the holographic principle is not universal and does not apply when extra dimensions are present; or, on the contrary, we take seriously the holographic claim in any number of dimensions, and our results are therefore evidence against the existence of extra dimensions. The four-dimensional Newton constant is denoted by G_{N} throughout the paper.

2. Ng and Van Dam GUP in four dimensions

An interesting GUP that satisfies the holographic principle in four dimensions has been proposed by Ng and van Dam [3], based on Wigner inequalities about distance measurements with clocks and light signals [6].

Suppose we wish to measure a distance l. Our measuring device is composed of a clock, a photon detector and a photon gun. A mirror is placed at the distance l which we want to measure and m is the mass of the system "clock + photon detector + photon gun". We call "detector" the whole system and let a be its size. Obviously, we suppose

$$a > r_{\mathrm{g}} \equiv \frac{2\,G_{\mathrm{N}}\,m}{c^2} = R_{\mathrm{S}}(m) \ , \tag{1}$$

which means that we are not using a black hole as a clock. Be Δx_1 the uncertainty in the position of the detector, then the uncertainty in the detector's velocity is

$$\Delta v = \frac{\hbar}{2\,m\,\Delta x_1} \ . \tag{2}$$

After the time $T = 2\,l/c$ taken by light to travel along the closed path detector–mirror–detector, the uncertainty in the detector's position (i.e. the uncertainty in the actual length of the segment l) has become

$$\Delta x_{\mathrm{tot}} = \Delta x_1 + T\,\Delta v = \Delta x_1 + \frac{\hbar\,T}{2\,m\,\Delta x_1} \ . \tag{3}$$

We can minimize Δx_{tot} by suitably choosing Δx_1, and we get

$$(\Delta x_{\mathrm{tot}})_{\mathrm{min}} = (\Delta x_1)_{\mathrm{min}} + \frac{\hbar T}{2\,m\,(\Delta x_1)_{\mathrm{min}}} = 2\left(\frac{\hbar\,T}{2\,m}\right)^{1/2} \ . \tag{4}$$

Since $T = 2\,l/c$, we have

$$(\Delta x_{\mathrm{tot}})_{\mathrm{min}} = 2\left(\frac{\hbar\,l}{m\,c}\right)^{1/2} \equiv \delta l_{\mathrm{QM}} \ . \tag{5}$$

This is a purely quantum mechanical result obtained for the first time by Wigner in 1957 [6]. From Eq. (5), it seems that we can reduce the error $(\Delta x_{\rm tot})_{\rm min}$ as much as we want by choosing m very large, since $(\Delta x_{\rm tot})_{\rm min} \to 0$ for $m \to \infty$. But, obviously, *here gravity enters the game.*

In fact, Ng and van Dam have also considered a further source of error, a gravitational error, besides the quantum mechanical one already addressed. Suppose the clock has spherical symmetry, with $a > r_{\rm g}$. Then the error due to curvature can be computed from the Schwarzschild metric surrounding the clock. The optical path from $r_0 > r_{\rm g}$ to a generic point $r > r_0$ is given by (see, for example, Ref. 7)

$$c\,\Delta t = \int_{r_0}^{r} \frac{d\rho}{1-\frac{r_{\rm g}}{\rho}} = (r-r_0) + r_{\rm g}\,\log\frac{r-r_{\rm g}}{r_0-r_{\rm g}}\ , \tag{6}$$

and differs from the "true" (spatial) length $(r-r_0)$. If we put $a = r_0$, $l = r$, the gravitational error on the measure of $(l-a)$ is thus

$$\delta l_{\rm C} = r_{\rm g}\,\log\frac{l-r_{\rm g}}{a-r_{\rm g}} \sim r_{\rm g}\,\log\frac{l}{a}\ , \tag{7}$$

where the last estimate holds for $l > a \gg r_{\rm g}$.

If we measure a distance $l \geq 2a$, then the error due to curvature is

$$\delta l_{\rm C} \geq r_{\rm g}\log 2 \simeq \frac{G_{\rm N} m}{c^2}. \tag{8}$$

Thus, according to Ng and van Dam the total error is

$$\delta l_{\rm tot} = \delta l_{\rm QM} + \delta l_{\rm C} = 2\left(\frac{\hbar\, l}{m\, c}\right)^{1/2} + \frac{G_{\rm N}\, m}{c^2}\ . \tag{9}$$

This error can be minimized again by choosing a suitable value for the mass of the clock, namely $m_{\rm min} = c(\hbar l)^{1/3}/G_{\rm N}$ and, inserting $m_{\rm min}$ in Eq. (9), we then have

$$(\delta l_{\rm tot})_{\rm min} = 3\,\left(\ell_{\rm p}^2\, l\right)^{1/3}\ . \tag{10}$$

The global uncertainty on l contains therefore a term proportional to $l^{1/3}$.

2.1. *Holographic properties*

We now see immediately the beauty of the Ng and van Dam GUP: it obeys the holographic scaling. In fact in a cube of size l the number of degrees of freedom is given by

$$N(V) = \left(\frac{l}{(\delta l_{\rm tot})_{\rm min}}\right)^3 = \left(\frac{l}{(\ell_{\rm p}^2\, l)^{1/3}}\right)^3 = \frac{l^2}{\ell_{\rm p}^2}\ , \tag{11}$$

as required by the holographic principle.

3. Models with n extra dimensions

We shall now generalize the procedure outlined in a previous section to a space-time with $4 + n$ dimensions, where n is the number of space-like extra dimensions [4]. The link between the gravitational constant $G_{\rm N}$ in four dimensions and the one in $4 + n$, henceforth denoted by $G_{(4+n)}$, of course depends on the model of space-time with extra dimensions that we consider. Models recently appeared in the literature mostly belong to two scenarios: (I) the Arkani-Hamed–Dimopoulos–Dvali (ADD) model [1], where the extra dimensions are compact and of size L; (II) the Randall–Sundrum (RS) model [2], where the extra dimensions have an infinite extension but are warped by a non-vanishing cosmological constant. A feature shared by (the original formulations of) both scenarios is that only gravity propagates along the n extra dimensions, while Standard Model fields are confined on a four-dimensional sub-manifold usually referred to as the *brane-world*.

In the ADD case the link between $G_{\rm N}$ and $G_{(4+n)}$ can be fixed by comparing the gravitational action in four dimensions with the one in $4+n$ dimensions. The space-time topology in such models is $\mathcal{M} = \mathcal{M}^4 \otimes \Re^n$, where $\mathcal{M}^4$ is the usual four-dimensional space-time and $\Re^n$ represents the extra dimensions of finite size L. From such comparison we obtain

$$G_{(4+n)} \sim G_{\rm N}\, L^n, \tag{12}$$

where we omit unimportant numerical factors.

The RS models are more complicated. It can be shown [2] that for $n = 1$ extra dimension we have $G_{(4+n)} = \sigma^{-1}\, G_{\rm N}$, where σ is the brane tension with dimensions of $length^{-1}$ in suitable units. The gravitational force between two point-like masses m and M on the brane is obtained by perturbative calculations, not immediately applicable to a non-perturbative structure such as a black hole. Therefore we shall consider only the ADD scenario in this paper (see Ref. 4 for more details).

4. Ng and Van Dam GUP in $4 + n$ dimensions

Ng and van Dam's derivation can be generalized to the case with n extra dimensions. The Wigner relation (5) for the quantum mechanical error is not modified by the presence of extra dimensions and we just need to estimate the error $\delta l_{\rm C}$ due to curvature.

We ought not to consider micro black holes created by the fluctuations ΔE in energy, as in Ref. 5, but we have rather to deal with (more or less) macroscopic clocks and distances. This implies that we have to distinguish four different cases: (1) $0 < L < r_g < a < l$; (2) $0 < r_{(4+n)} < L < a < l$; (3) $0 < r_{(4+n)} < a < L < l$; (4) $0 < r_{(4+n)} < a < l < L$; where $r_{(4+n)}$ is the Schwarzschild radius of the detector in $4+n$ dimensions, and of course $r_{\rm g} = r_{(4)}$. The curvature error will be estimated (as before) by computing the optical path from $a \equiv r_0$ to $l \equiv r$. Of course, we will use a metric which depends on the relative size of L with respect to a and l, that is the usual four-dimensional Schwarzschild metric in the region $r > L$, and the $4+n$ dimensional Schwarzschild solution in the region $r < L$ (where the extra dimensions play an actual role).

In cases (1) and (2) the length of the optical path from a to l can be obtained using just the four-dimensional Schwarzschild solution and the result is given by Eq. (10).

In cases (3) and (4) we instead have to use the Schwarzschild solution in $4+n$ dimensions [11],

$$ds^2 = -\left(1 - \frac{C}{r^{n+1}}\right) c^2 dt^2 + \left(1 - \frac{C}{r^{n+1}}\right)^{-1} dr^2 + r^2 d\Omega_{n+2}^2 \ , \tag{13}$$

at least for part of the optical path. In the above,

$$C = \frac{16\,\pi\, G_{(4+n)}\, m}{(n+2)\, A_{n+2}\, c^2} \ , \tag{14}$$

and A_{n+2} is the area of the unit $(n+2)$-sphere, that is

$$A_{n+2} = \frac{2\,\pi^{\frac{n+3}{2}}}{\Gamma\left(\frac{n+3}{2}\right)} \ . \tag{15}$$

Besides, we note that, for $n = 0$,

$$C = \frac{2\, G_{\rm N}\, m}{c^2} = r_{\rm g} \ , \tag{16}$$

that is, C coincides in four dimensions with the Schwarzschild radius of the detector. The $4+n$ dimensional Schwarzschild horizon is located where $(1 - C/r^{n+1}) = 0$, that is at

$$r = C^{1/(n+1)} \equiv r_{(4+n)}. \tag{17}$$

Since measurements can be performed only on the brane, to the uncertainty Δx in position we can still associate an energy given by $\hbar\, c/(2\,\Delta x)$. The corresponding Schwarzschild radius is now given by Eq. (17) with $m = \Delta E/c^2$ and the critical length such that $\Delta x = r_{(4+n)}$ is the Planck length in $4+n$ dimensions,

$$\Delta x \simeq \left(\ell_{\rm p}^2\, L^n\right)^{\frac{1}{n+2}} \equiv \ell_{(4+n)} \ . \tag{18}$$

In case (3) we obtain the length of the optical path from a to l by adding the optical path from a to L and that from L to l. We must use the solution in $4+n$ dimensions for the first part, and the four-dimensional solution for the second part of the path,

$$c\,\Delta t = \int_a^L \left(1 + \frac{C}{r^{n+1} - C}\right) dr + \int_L^l \left(1 + \frac{r_{\mathrm{g}}}{r - r_{\mathrm{g}}}\right) dr. \tag{19}$$

It is not difficult to show that from $r_{(4+n)} < L$ [which holds in cases (3) and (4)] we can infer

$$r_g < r_{(4+n)} < L\ . \tag{20}$$

We suppose $a^{n+1} \gg C = r_{(4+n)}^{n+1}$, that is $a \gg r_{(4+n)}$, so that we are not doing measures inside a black hole. Then $r_g < r_{(4+n)} \ll a < L < l$.

In case (4), the optical path from a to l can be obtained by using simply the Schwarzschild solution in $4+n$ dimensions. We get

$$c\,\Delta t = \int_a^l \left(1 + \frac{C}{r^{n+1} - C}\right) dr = (l-a) + C\int_a^l \frac{dr}{r^{n+1} - C}\ . \tag{21}$$

Also here we suppose, as before, that $a^{n+1} \gg C = r_{(4+n)}^{n+1}$, that is $a \gg r_{(4+n)}$ (i.e. our clock is not a black hole).

If the distance we are measuring is, at least, of the size of the clock ($l \geq 2\,a$), we can minimize δl_{tot} with respect to m in perfect analogy with the previous calculation and we obtain that the total minimum error (quantum mechanical + curvature) can be written in both cases in the form

$$(\delta l_{\mathrm{tot}})_{\mathrm{min}} = \alpha(n)\left(\frac{\ell_{(4+n)}^{n+2}\, l}{a^n}\right)^{1/3}\ , \tag{22}$$

where $\alpha(n)$ is an unimportant numerical factor (for detailed calculations see Ref. 4). Note that, for $n = 0$, Eq. (22) yields the four-dimensional error given in Eq. (10).

4.1. *Holographic properties*

We finally examine the holographic properties of Eq. (22) for the GUP of Ng and van Dam type in $4+n$ dimensions.

Since we are just interested in the dependence of $N(V)$ on l and the basic constants, we can write

$$(\delta l_{\mathrm{tot}})_{\mathrm{min}} \sim \left(\frac{\ell_{(4+n)}^{n+2}\, l}{a^n}\right)^{1/3}\ . \tag{23}$$

We then have that the number of degrees of freedom in the volume of size l is

$$N(V) = \left(\frac{l}{(\delta l_{\rm tot})_{\rm min}}\right)^{3+n} = \left(\frac{l^2\, a^n}{\ell_{\rm p}^2\, L^n}\right)^{1+\frac{n}{3}} \ , \tag{24}$$

and the holographic counting holds in four-dimensions ($n = 0$) but is lost when $n > 0$. In fact we do not get something as

$$N(V) = \left(\frac{l}{\ell_{(4+n)}}\right)^{2+n} \ , \tag{25}$$

as we would expect in $4 + n$ dimensions. Even if we take the ideal case $a \sim \ell_{(4+n)}$ we get

$$N(V) = \left(\frac{l}{\ell_{(4+n)}}\right)^{2\left(1+\frac{n}{3}\right)} \ , \tag{26}$$

and the holographic principle does not hold for $n > 0$.

5. Concluding remarks

In the previous Sections, we have shown that the holographic principle seems to be satisfied only by uncertainty relations in the version of Ng and van Dam *and* for $n = 0$. That is, only in four dimensions we are able to formulate uncertainty principles which predict the same number of degrees of freedom per spatial volume as the holographic counting. This could be evidence for questioning the existence of extra dimensions.

Moreover, such an argument based on holography could also be used to support the compactification of string theory down to four dimensions, given that there seems to be no firm argument which forces the low energy limit of string theory to be four-dimensional (except for the obvious observation of our world). Another interesting possibility could be that the Schwarzschild solution in $4 + n$ dimensions should be modified, if we want to preserve both *holography* and *extra dimensions*.

We should also say that the cases (3) and (4) of Section 4 do not have a good probability to be experimentally realized since, if there are extra spatial dimensions, their size must be shorter than 10^{-1} mm [8]. Therefore, cases (1) and (2) of Section 4 are more likely going to be tested in future experiments.

References

1. N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B 429**, 263 (1998); Phys. Rev. D **59**, 0806004 (1999); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B 436**, 257 (1998).
2. L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); Phys. Rev. Lett. **83**, 4690 (1999).
3. Y.J. Ng and H. van Dam, Mod. Phys. Lett. **A 9**, 335 (1994); Mod. Phys. Lett. **A 10**, 2801 (1995); Phys. Lett. **B 477**, 429 (2000); Y.J. Ng, Phys. Rev. Lett. **86**, 2946 (2001).
4. F. Scardigli and R. Casadio, Class. Quantum Grav. **20** (2003) 3915.
5. F. Scardigli, Phys. Lett. **B 452**, 39 (1999).
6. E.P. Wigner, Rev. Mod. Phys. **29**, 255 (1957); H. Salecker and E.P. Wigner, Phys. Rev. **109**, 571 (1958).
7. L.D. Landau and E.M. Lifshitz, *The classical theory of fields*, Pergamon Press, Oxford, 1975.
8. C.D. Hoyle et al., Phys. Rev. Lett. **86**, 1418 (2001).
9. S.B. Giddings, E. Katz and L. Randall, JHEP **0003**, 023 (2000); J. Garriga and T. Tanaka, Phys. Rev. Lett. **84**, 2778 (2000).
10. P.C. Argyres, S. Dimopoulos and J. March-Russell, Phys. Lett. **B 441**, 96 (1998).
11. R.C. Myers and M.J. Perry, Annals of Physics **172**, 304 (1986).