Singularities and braneworlds in quantum cosmology

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Plan of the talk

- 1. Physical motivations
- 2. Boundary conditions
- 3. Eigenvalue condition for scalar modes
- 4. Four zeta-functions for scalar modes
- 5. Non-singular one-loop wave function
- 6. Functional integrals for brane cosmology
- 7. Open problems

1. Physical motivations

- (i) Functional integrals and space-time view. Is the Universe still singular in quantum cosmology?
- (ii) Early universe from quantum physics; wave function of the Universe and Hartle-Hawking quantum cosmology.

1. Other physical motivations

- (iii) The whole set of physical laws from invariance principles.
- (iv) Spectral theory; functional determinants in one-loop quantum theory; first corrections to classical dynamics.

2. Boundary conditions

- Unified scheme for Maxwell, YM, GR:

(1) $$\left[\pi \mathcal{A}\right]_{\mathcal{B}} = 0,$$

(2) $$\left[\Phi(\mathcal{A})\right]_{\mathcal{B}} = 0,$$

(3) $$\left[\varphi\right]_{\mathcal{B}} = 0.$$

Here π is a projector acting on the gauge field $\mathcal{A}$, Φ is the gauge-fixing functional, φ the ghost field.

Gauge-invariant boundary data

- Both (1) and (2) are preserved under infinitesimal gauge transformations provided that the ghost obeys homogeneous Dirichlet conditions as in (3). For gravity, we choose the gauge-fixing so as to have an operator P of Laplace type in the Euclidean theory.

3. Eigenvalue condition for scalar modes

- On the Euclidean 4-ball, we expand metric perturbations h in terms of scalar, transverse vector, transverse-traceless tensor harmonics on the 3-sphere. For vector, tensor and ghost modes, boundary conditions reduce to Dirichlet or Robin. For scalar modes, one finds eventually the eigenvalues E=x**2 from the roots x of

3. Bessel functions and their derivative

(4) $$J_n'(x) \pm \frac{n}{x} J_n(x) = 0,$$

(5) $$J_n'(x) + \left(-\frac{x}{2} \pm \frac{n}{x}\right) J_n(x) = 0.$$

Note that both x and $-x$ solve the same equation.

4. Four spectral zeta-functions for scalar modes

(6) $$\zeta^{\pm}_{A,B}(s) \equiv \frac{(\sin \pi s)}{\pi} \sum_{n=3}^{\infty} n^{-(2s-2)} \int_0^{\infty} dz \; \frac{\frac{\partial}{\partial z} \log F^{\pm}_{A,B}(zn)}{z^{2s}}$$

where (here $\beta_+ \equiv n, \beta_- \equiv n+2$)

(7) $$F^{\pm}_{A}(zn) \equiv z^{-\beta_{\pm}} \Big(zn I_n'(zn) \pm n I_n(zn) \Big),$$

Modified Bessel functions

(8) $$F_B^{\pm}(zn) \equiv z^{-\beta_\pm}\left(znI_n'(zn) + \left(\frac{z^2n^2}{2} \pm n\right) I_n(zn)\right).$$

Regularity at the origin is easily proved in the elliptic sectors, corresponding to $\zeta_A^{\pm}(s)$ and $\zeta_B^{-}(s)$.

5. Non-singular one-loop wave function

We define $\tau \equiv (1+z^2)^{-1/2}$ and consider the uniform asymptotic expansion

(9) $$F_B^+(zn) \sim \frac{e^{n\eta(\tau)}}{h(n)\sqrt{\tau}} \frac{(1-\tau^2)}{\tau} \left(1 + \sum_{j=1}^{\infty} \frac{r_{j,+}(\tau)}{n^j} \right).$$

On splitting $\int_0^1 d\tau = \int_0^\mu d\tau + \int_\mu^1 d\tau$ with μ small, we get an asymptotic expansion of the l.h.s. by writing, *in the first interval* on the r.h.s.,

5. Uniform asymptotics

(10) $$\log\left(1+\sum_{j=1}^{\infty}\frac{r_{j,+}(\tau)}{n^j}\right)\sim\sum_{j=1}^{\infty}\frac{R_{j,+}(\tau)}{n^j},$$

and then computing

(11) $$C_j(\tau)\equiv\frac{\partial R_{j,+}}{\partial\tau}=(1-\tau)^{-j-1}\sum_{a=j-1}^{4j}K_a^{(j)}\tau^a.$$

5. Spectral cancellation

Remarkably, by virtue of the identity

(12) $$g(j) \equiv \sum_{a=j}^{4j} \frac{\Gamma(a+1)}{\Gamma(a-j+1)} K_a^{(j)} = 0,$$

which holds $\forall j = 1, \ldots, \infty$, we find

(13) $$\lim_{s \to 0} s\zeta_B^+(s) = \frac{1}{6} \sum_{a=3}^{12} a(a-1)(a-2) K_a^{(3)} = 0,$$

5. Regularity at the origin

and

$$\zeta_B^+(0) = \frac{5}{4} + \frac{1079}{240} - \frac{1}{2}\sum_{a=2}^{12}\omega(a)K_a^{(3)}$$

(14)

$$+ \sum_{j=1}^{\infty} f(j)g(j) = \frac{296}{45},$$

where

5. Log, Gamma and Poli-Gamma

$$\begin{aligned}\omega(a) &\equiv \frac{1}{6}\frac{\Gamma(a+1)}{\Gamma(a-2)}\Big[-\log(2)\\ &- \frac{(6a^2-9a+1)}{4}\frac{\Gamma(a-2)}{\Gamma(a+1)}\\ &+ 2\psi(a+1)-\psi(a-2)-\psi(4)\Big],\end{aligned} \tag{15}$$

5. Three goals at once

$$f(j) \equiv \frac{(-1)^j}{j!}\Big[-1-2^{2-j} + \zeta_R(j-2)(1-\delta_{j,3}) + \gamma\delta_{j,3}\Big]. \tag{16}$$

Equation (12) achieves 3 goals:

(i) Vanishing of $\log(2)$ coefficient in (14)
(ii) Vanishing of $\sum_{j=1}^{\infty} f(j)g(j)$ in (14)
(iii) Regularity at the origin of ζ_B^+

5. Cross-check

To cross-check our analysis, we evaluate $r_{j,+}(\tau) - r_{j,-}(\tau)$ and hence obtain $R_{j,+}(\tau) - R_{j,-}(\tau)$ for all j. Only $j = 3$ contributes to $\zeta_B^{\pm}(0)$, and we find

5. Regularity at the origin

$$
\begin{aligned}
\zeta_B^{+}(0) &= \zeta_B^{-}(0) \\
&- \frac{1}{24}\sum_{l=1}^{4}\frac{\Gamma(l+1)}{\Gamma(l-2)}\left[\psi(l+2)-\frac{1}{(l+1)}\right]\kappa_{2l+1}^{(3)} \\
&= \frac{206}{45}+2=\frac{296}{45}, \qquad (17)
\end{aligned}
$$

5. Spectral coefficients

in agreement with Eq. (14), where $\kappa_{2l+1}^{(3)}$ are the four coefficients on the right-hand side of

$$\frac{\partial}{\partial\tau}(R_{3,+} - R_{3,-}) = (1-\tau^2)^{-4}\Big(80\tau^3 - 24\tau^5 + 32\tau^7 - 8\tau^9\Big).$$

5. Vanishing 1-loop wavefunction

Spectral cancellation:

(19)
$$\sum_{l=1}^{4} \frac{\Gamma(l+1)}{\Gamma(l-2)} \kappa^{(3)}_{2l+1} = 0,$$

which is a particular case of

(20)
$$\sum_{a=a_{\min}(j)}^{a_{\max}(j)} \frac{\Gamma((a+1)/2)}{\Gamma((a+1)/2-j)} \kappa^{(j)}_{a} = 0.$$

Full $\zeta(0) = \frac{142}{45}$ leads to vanishing 1-loop wavefunction!

6. Functional integrals for brane cosmology

In the braneworld picture, branes are timelike surfaces embedded into bulk space-time. Their normal vector N^A is therefore spacelike with respect to the bulk metric G_{AB}, i.e.

(21) $$G(N,N) = G_{AB}N^A N^B = N_C N^C > 0.$$

6. Towards the effective action

The action functional reads as

(22) $$S = S_4[g_{\alpha\beta}(x)] + S_5[G_{AB}(X)],$$

while the effective action is given by

(23) $$\mathrm{e}^{\mathrm{i}\Gamma} = \int DG_{AB}(X)\mathrm{e}^{\mathrm{i}S} \times \mathrm{g.f.\ term},$$

6. Expansion of bulk and brane metrics

(24) $$S_{\mathrm{g.f.}} = S_4 + S_5 + \frac{1}{2}F^A \widetilde{C}_{AB} F^B + \frac{1}{2}\chi^\mu C_{\mu\nu}\chi^\nu .$$

For example, we may expand bulk and brane metrics according to

(25) $$G_{AB} = G^{(0)}_{AB} + H_{AB},\ g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + h_{\alpha\beta},$$

6. de Donder gauges

and impose de Donder gauges, i.e.

(26) $$F_A = \nabla^B H_{AB} - \frac{1}{2}\nabla_A H,$$

(27) $$\chi_\mu = \nabla^\nu h_{\mu\nu} - \frac{1}{2}\nabla_\mu h.$$

The bulk and brane ghost operators are then

(28) $$Q^A_{\ B} = \delta^A_{\ B}\Box_5 + R^A_{\ B}, \ \ J^\mu_{\ \nu} = \delta^\mu_{\ \nu}\Box_4 + R^\mu_{\ \nu}.$$

6. Vector fields on the space of histories

In general, there exist vector fields R_B, R_ν on the space of histories s.t.

(29) $$R_B S_5 = 0, \; R_\nu S_4 = 0,$$

(30) $$[R_B, R_D] = C^A{}_{BD} R_A, \; [R_\mu, R_\nu] = C^\lambda{}_{\mu\nu} R_\lambda.$$

6. 5-D and 4-D diffeomorphisms

The components of such vector fields generate 5-D and 4-D diffeomorphisms, respectively, i.e.

(31) $$(L_\xi G)^a = R^a{}_A \xi^A, \ (L_\xi g)^i = R^i{}_\mu \xi^\mu,$$

where $G_{AB}(X) \to G^a, \ g_{\alpha\beta}(x) \to g^i$.

6. Bulk and brane ghost operators

The bulk and brane ghost operators are therefore, in general,

(32) $$Q^A_{\ B} = R_B F^A = F^A_{\ ,a} R^a_{\ B},$$

(33) $$J^\mu_\nu = R_\nu \chi^\mu = \chi^\mu_{\ ,i} R^i_{\ \nu}.$$

6. Gauge-fixing terms

In Eq. (23) for the effective action, the gauge-fixing term in the integrand is the product of

$$e^{\frac{\mathrm{i}}{2}F^A \widetilde{C}_{AB} F^B} \ \mathrm{Det}\,{}_D Q^A_{\ B}$$

with

$$e^{\frac{\mathrm{i}}{2}\chi^\mu C_{\mu\nu}\chi^\nu} \ \det J^\mu_{\ \nu}.$$

6. Full bulk integration

The full bulk integration means

$$DG_{AB}(X) = \int dg_{\alpha\beta}(x) \int_{G_{AB}[\partial M]=g_{\alpha\beta}(x)} DG_{AB}(X).$$

Thus, one first evaluates the cosmological wave function of the bulk space-time, i.e.

6. Wave function of the bulk

(34) $$\psi_{\mathrm{Bulk}} = \int_{G_{AB}[\partial M]=g_{\alpha\beta}} \mu(G_{AB}, S, T)\mathrm{e}^{\mathrm{i}\widetilde{S}_5},$$

where

(35) $$\widetilde{S}_5 = S_5[G_{AB}] + \frac{1}{2}F^A\widetilde{C}_{AB}F^B + S_A Q^A_B T^B.$$

6. Full effective action

Eventually, the effective action results from

(36) $$e^{i\Gamma} = \int \mu(g_{\alpha\beta}, \rho, \sigma) e^{i\widetilde{S}_4} \psi_{\mathrm{Bulk}},$$

where

(37) $$\widetilde{S}_4 = S_4 + \frac{1}{2}\chi^{\mu} C_{\mu\nu} \chi^{\nu} + \rho_{\mu} J^{\mu}_{\nu} \sigma^{\nu}.$$

6. Bulk BRST transformations

This scheme is invariant under infinitesimal BRST transformations

(38) $$\delta G^a = R^a{}_A T^A \delta\Lambda,$$

(39) $$\delta S_A = \widetilde{C}_{AB} F^B \delta\Lambda,$$

(40) $$\delta T^A = -\frac{1}{2} C^A_{BD} T^B T^D \delta\Lambda,$$

where $T^A T^B + T^B T^A = 0,\ T^A \delta\Lambda + (\delta\Lambda) T^A = 0$,

6. Brane BRST transformations

as well as

(41) $$\delta g^i = R^i{}_\mu \sigma^\mu \delta\lambda,$$

(42) $$\delta \rho_\mu = C_{\mu\nu} \chi^\nu \delta\lambda,$$

(43) $$\delta\sigma^\mu = -\frac{1}{2} C^\mu{}_{\nu\zeta} \sigma^\nu \sigma^\zeta \delta\lambda,$$

where $\sigma^\nu \sigma^\zeta + \sigma^\zeta \sigma^\nu = 0,\ \sigma^\nu \delta\lambda + (\delta\lambda)\sigma^\nu = 0$.

6. Action for brane models

In general, for a brane model:

$$S[\phi] = \frac{1}{2}\int_B \phi(X)F(\nabla)\phi(X)dX$$

(44) $$+ \int_b \left(\frac{1}{2}\varphi(x)k(\partial)\varphi(x) + j(x)\varphi(x)\right),$$

where $\varphi(x)$ denotes boundary values of bulk fields $\phi(X)$.

6. Oblique boundary conditions

The action $S[\phi]$ gives rise to generalized Neumann B.C.

(45) $$\left[(W(\nabla) + k(\partial))\phi\right]_{\partial B} = 0,$$

where $W(\nabla)$ is the Wronskian operator on ∂B.

6. Duality of boundary-value problems

The Dirichlet and Neumann B.V.P. are “dual” in that

$$\begin{aligned} \Gamma_{1-\mathrm{loop}} &= \frac{1}{2}\mathrm{Tr}_N \log F \\ &= \frac{1}{2}\mathrm{Tr}_D \log F + \frac{1}{2}\mathrm{tr}\log P^{\mathrm{brane}}, \end{aligned} \qquad (46)$$

where $P^{\mathrm{brane}} = -[WG_DW]_b(x,y) + k(\partial)\delta(x,y)$, G_D being the Dirichlet Green function of the operator $F(\nabla)$ in the bulk.

7. Open problems

- (i) Are our spectral cancellations a peculiar property of the Euclidean 4-ball, or can they be extended to more general Riemannian manifolds with boundary?
- (ii) What is the deeper underlying reason for finding \delta \zeta(0)=2 for some scalar sectors? Can we foresee a geometrical or topological or group-theoretical origin of this result?

7. More open problems

- (iii) Can we say that our positive \zeta(0) value for pure gravity engenders a quantum avoidance of the cosmological singularity at one-loop level? Does the result remain true in higher-loop calculations or on using other regularization techniques for the one-loop correction?

7. Yet other open problems

- (iv) Is the whole scheme relevant for AdS/CFT, in light of a profound link between AdS/CFT and the Hartle-Hawking wave function of the Universe?
- (v) How to build and apply the background field method for quantum brane cosmology?

References

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