La rappresentazione tomografica in cosmologia classica e quantistica

Cosimo Stornaiolo
INFN – Sezione di Napoli

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V.I Man’ko, G. Marmo and C.S.

1. “Radon Transform of the Wheeler-De Witt equation and tomography of quantum states of the universe” Gen. Relativ. Gravit. (2005) 37: 99–114
2. “Cosmological dynamics in tomographic probability representation” (gr-qc/0412091) published on GRG

S. Capozziello, V.I Man’ko, G. Marmo and C.S.

1. “ Tomographic probability representation in cosmological problem of minisuperspace evolution with Noetherian symmetries. (in preparation)

Motivations for this work

- The role of the Quantum Potential at early times in Cosmology
- The initial conditions problem in Quantum Cosmology
- Why Tomographic approach to Quantum Cosmology?
- Phenomenological approach to Quantum Cosmology
- Evolution of the universe from a quantum to classical: Noether symmetries and decoherence
- Quantum fluctuations and classical perturbations

The initial condition problem

- The initial conditions of the universe can be considered as a fundamental law in Quantum Cosmology
- Do we have to postulate these conditions?
- Or can we have a phenomenological approach to this problem?
- Differently from the wave function approach we can deal classical and quantum cosmology with the *same* variable.
- The difference is just in the initial conditions

Phenomenological Quantum Cosmology

- Our approach appears to be promising, because tomograms are in principle measurable
- Classical and Quantum tomograms are comparable
- In future work we shall need to define the measurement of a cosmological tomogram
- This will enable us to study the initial conditions problem from a phenomenological point of view
- Moreover we hope to be able to distinguish the quantum evolution from the classical one.

What we can know from observations?

We can expect that observations put some constraints on the present tomogram (and consequently to the initial conditions) , we must use observations of

- Entropy $S/k = -Tr(\rho \log \rho) \approx 10^{80} << 10^{120}$
- Cosmic background radiation fluctuations (WMAP)

$$\frac{\delta T}{T} \leq 10^{-5}$$

- Approximate homogeneity and isotropy
- Perturbation spectrum and structures formation (SDSS and comparison between WMAP and SDSS)

Wheeler-de Witt equation in Quantum Gravity

- canonical approach
- equation in the space of three dimensional metrics

$$x \rightarrow h_{ij} \qquad \text{spatial metric}$$

$$\left(-G_{ijkl} \frac{\delta^2}{\delta h_{ij} \delta h_{kl}} - {}^3R(h) h^{1/2} + 2\Lambda h^{1/2} \right) \psi\left(h_{ij}\right) = 0$$

G_{ijkl} metric the space of three geometries (superspace)

Quantum Cosmology

1. Minisuperspace: considering only homogeneous metrics
2. cosmological models as a point particles
3. working with a finite number of degrees of freedom
4. violates the uncertainty principle fixing contemporarily a zero infinite variables and their momenta

Wheeler-de Witt equation in Quantum Cosmology

Here is an example of a Wheeler-deWitt equation in the space of homogeneous and isotropic metrics

$$\frac{1}{2}\left\{\frac{1}{a^p}\frac{d}{da}a^p\frac{d}{da}-a^2+\Lambda a^4\right\}\psi(\mathrm{a})=0$$

a model with cosmological constant and no matter fields is considered, the exponent ‘’p’’ reflects the ambiguity of the theory in fixing the order of operators.

For large values of the expansion factor a the solution is

$$\psi(\mathrm{a})\approx\cos\frac{\mathrm{Ha}^3}{3}$$

Quantum Mechanics

- Uncertainty principle $\Delta p \Delta q \geq \hbar / 2$
- Schrödinger Equation

$$\hat{H}\psi(\mathrm{x},\mathrm{t}) = i\hbar \frac{\partial\psi(\mathrm{x},\mathrm{t})}{\partial t}$$

- Observables and measurements
- Physical interpretation

Please note: **Not Copenhagen** interpretation of this quantum theory

The Tomographic Approach to Quantum Mechanics

- Quantum mechanics without wave function and density matrix.
- New formulation of Q. M. based on the ”probability representation” of quantum states.
- Introduction of the marginal probability functions (tomograms)
- They contain exactly the same informations of the wave functions (or the density matrix or the Wigner distribution)
- But in this case we deal with the evolution of a measurable quantity
- whose evolution is classical or quantum depending on the initial conditions, that can be classical or quantum

The tomographic map

Density $\rho(\mathrm{x}, \mathrm{x}', \mathrm{t}) = \psi(\mathrm{x}, \mathrm{t})\,\psi^*(x', t)$

Tomographic map

$$\mathrm{w}(\mathrm{X}, \mu, \nu, \mathrm{t}) = \frac{1}{2\pi\,|\,\nu\,|}\int \rho(\mathrm{y}, \mathrm{y}', \mathrm{t}) \times \exp\left[\mathrm{i}\,\frac{\mu(\mathrm{y}^2 - y'^2)}{2\nu} - i\,\frac{X(y - y')}{\nu}\right] dy\,dy'$$

or in of the Wigner distribution function

$$w(X, \mu, \nu, t) = \tfrac{1}{2\pi}\int W(q, p, t)\,\delta(X - \mu q - \nu p)\,dq\,dp$$

Relation between tomograms and wave function

Tomograms contain the same information of wave functions, they are defined by considering the following trasformation

$$w(X,\mu,\nu) = \frac{1}{2\pi|\nu|}\left|\int \psi(y)\exp\left(\frac{i\mu}{2\nu}y^2 - \frac{iX}{\nu}y\right)dy\right|^2$$

where the variable

$$X = \mu q + \nu p$$

is a "coordinate" transformation on phase space

$$\mu = s\cos\varphi \qquad \nu = s\sin\phi$$

Properties of the tomograms

1. They are non negative $w(X,\mu,\nu) \geq 0$

2. $$\int w(X,\mu,\nu)\,\mathrm{d}X = 1$$

The Classical Tomogram

The classical tomogram is obtained by substituting the Wigner function with the solution of the classical Liouville equation.

However classical and quantum tomograms are defined on the same space and therefore can be compared.

The Tomogram Equation

Alternative to the Schroedinger equation

$$i\dot{\psi} = H\psi$$

we find the equation for the tomogram

$$\dot{w} - \mu \frac{\partial}{\partial \nu} w + 2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \frac{\partial^{2n+1} V}{\partial x^{2n+1}} \left(\frac{\nu}{2} \frac{\partial}{\partial X} \right)^{2n+1} (-1)^{2n+1} w = 0$$

Wheeler-de Witt equation in Quantum Cosmology

Let us recall the Wheeler-deWitt equation presented previously

$$\frac{1}{2}\left\{\frac{1}{a^p}\frac{d}{da}a^p\frac{d}{da} - a^2 + \Lambda a^4\right\}\psi(\mathrm{a}) = 0$$

We can re-express it for the matrix density

$$\rho(a,a') = \psi(a)\psi^*(a')$$

Transformation relations

And using the following relations

$$\rho(x, x') \rightarrow W(X, \mu, \nu)$$

$$x \rightarrow -\left(\frac{\partial}{\partial X}\right)^{-1} \frac{\partial}{\partial \mu} + \frac{i}{2}\nu \frac{\partial}{\partial X}$$

$$x' \rightarrow -\left(\frac{\partial}{\partial X}\right)^{-1} \frac{\partial}{\partial \mu} - \frac{i}{2}\nu \frac{\partial}{\partial X}$$

$$\frac{\partial}{\partial x} \rightarrow \frac{1}{2}\mu \frac{\partial}{\partial X} - i\left(\frac{\partial}{\partial X}\right)^{-1} \frac{\partial}{\partial \nu}$$

$$\frac{\partial}{\partial x'} \rightarrow \frac{1}{2}\mu \frac{\partial}{\partial X} + i\left(\frac{\partial}{\partial X}\right)^{-1} \frac{\partial}{\partial \nu}$$

$$a = \exp x$$

The tomogram equation corresponding to the Wheeler de Witt equation

- We are obtain the corresponding equation for tomograms in quantum cosmology (see the preceding example)

$$\left\{ \mathrm{Im}\left[\exp\left[2\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial \mu} + i\nu\frac{\partial}{\partial X}\right]\left(\frac{1}{2}\mu\frac{\partial}{\partial X} - i\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial \nu}\right)^{2}\right] + \right.$$

$$(p-1)\,\mathrm{Im}\left[\exp\left[2\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial \mu} + i\nu\frac{\partial}{\partial X}\right]\left(\frac{1}{2}\mu\frac{\partial}{\partial X} - i\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial \nu}\right)\right]$$

$$\left. -2\,\mathrm{Im}\left[\exp\left[2\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial \mu} + i\nu\frac{\partial}{\partial X}\right] - \Lambda\exp - 4\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial \mu} + 2i\nu\frac{\partial}{\partial X}\right]\right\} w(X,\mu,\nu) = 0$$

$$\left(\frac{\partial}{\partial X}\right)^{-1} f(x) = \int \frac{\tilde{f}(k)e^{ikX}}{ik}$$

Cosmological metric

- Homogeneous and isotropic metric

$$ds^2 = -c^2dt^2 + \frac{a^2(t)}{1-kr^2}[dx^2 + dy^2 + dz^2]$$

$$k = +1$$
$$k = 0$$
$$k = -1$$

- In conformal time

$$d\eta = \frac{dt}{a(t)} \qquad ds^2 = a^2(\eta)\left[-c^2d\eta^2 + \frac{dx^2 + dy^2 + dz^2}{1-kr^2}\right]$$

Classical cosmological equations

- Friedmann equations

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho$$

$$P = (\gamma - 1)\rho$$

$$\rho = \rho_0 \frac{a^{\alpha}{}_0}{a^{\alpha}}$$

$$\ddot{a} = -\frac{4}{3}\pi G(\rho + 3P)$$

$$\alpha = \begin{cases} 3 : \gamma = 1 \\ 4 : \gamma = 4/3 \end{cases}$$

Cosmological models as harmonic oscillators

Let us make in a homogeneous and isotropic model in conformal time the change of variables

$$z = a^{\chi} \qquad \chi = \frac{3}{2}\gamma - 1$$

The evolution cosmological equation takes the form

$$z'' + k\chi^2 z = 0$$

$$k = +1$$
$$k = 0$$
$$k = -1$$

Cosmological models as harmonic oscillators (2)

In a similar way for cosmological models with a fluid and a cosmological constant one obtains (in cosmic time) putting

$$a = z^{\sigma} \qquad \chi = \tfrac{3}{2}\gamma - 1 \qquad \sigma = 1 - \chi$$

$$\ddot{z} = \frac{\chi \Lambda}{\sigma} z + \frac{k\chi}{z^{\sigma}}$$

If k=0 we have again the ‘’harmonic oscillator)

Tomographic equation for a harmonic oscillator

An useful equation is the harmonic oscillator equation for the tomograms, which is the same for classical and quantum tomograms

$$\frac{\partial W}{\partial t} - \mu \frac{\partial W}{\partial \nu} + \omega^2 \nu \frac{\partial W}{\partial \mu} = 0$$

Uncertainty Relations for tomograms

The uncertainty relation is

$$\left[\int w(X,1,0)X^2dX - \left\{\int w(X,1,0)XdX\right\}^2\right]\times$$

$$\left[\int w(X,0,1)X^2dX - \left\{\int w(X,0,1)XdX\right\}^2\right]\geq\frac{1}{4}$$

Propagators in the tomographic approach

- The evolution of a tomogram can be described by the transition probability

$$\Pi(X, \mu, \nu, t, X', \mu', \nu', t_0)$$

with the equation

$$W(X,\mu,\nu,t) = \int \Pi(X,\mu,\nu,t,X',\mu',\nu',t_0)\, W(X',\mu',\nu',t_0)\, dX'\, d\mu'\, d\nu'$$

Evolution of a tomogram of the universe

- The transition probility Π satisfies the following equation

$$\frac{\partial \Pi}{\partial t} - \mu \frac{\partial \Pi}{\partial \nu} + \omega^2 \nu \frac{\partial \Pi}{\partial \mu} = \delta(\mu - \mu')\delta(\nu - \nu')\delta(X - X')\delta(t - t_0)$$

Solutions for the transition probabilities

In the minisuperspace considered in this talk, the transition probabilities are

$$\Pi^{osc}(X,\mu,\nu,t,X',\mu',\nu',t_0) = \delta(X - X')\,\delta(\mu'-\mu\cos\omega(t - t_0) + \nu\sin\omega(t - t_0))$$
$$\times\,\delta\left(\nu'-\nu\cos\omega(t - t_0) - \frac{\mu}{\omega}\sin\omega(t - t_0)\right)$$

$$\Pi^{free}(X,\mu,\nu,t,X',\mu',\nu',t_0) = \delta(X - X')\,\delta(\mu'-\mu)\,\delta(\nu'-\nu - \mu(t - t_0))$$

$$\Pi^{rep}(X,\mu,\nu,t,X',\mu',\nu',t_0) = \delta(X - X')\,\delta(\mu'-\mu\cosh\omega(t - t_0) - \nu\sinh\omega(t - t_0)$$
$$\times\,\delta\left(\nu'-\nu\cosh\omega(t - t_0) - \frac{\mu}{\omega}\sinh\omega(t - t_0)\right)$$

Transition from quantum to classical evolution of the universe

- No boundary condition (Hartle-Hawking)
- Tunneling from nothing (Vilenkin)
- Hartle criterion: looking for peaks in the wave function
- Noether symmetries and oscillating behaviour: the emergence of a Noether symmetry gives to classical trajectories (Capozziello and Lambiase)
- Noether symmetries satisfy the Hartle criterion
- A test for theories alternative to GR

Noether symmetries

$$L_X \mathcal{L} = \alpha^i(q)\frac{\partial \mathcal{L}}{\partial q^i} + \dot{\alpha}^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = 0$$

$$|\,\psi >= \sum_{j=1}^{m} e^{i\Sigma_j Q^j} \;|\,\chi(Q^l) >$$

$$m < l \leq n$$

Semiclassical limit for QC

$$(\frac{1}{m_P^2}\nabla^2 - m_P^2 U)\Psi[h_{ij}(x),\phi] = 0$$

In semiclassical limit

$$\Psi[h_{ij}(x),\phi] \approx e^{im_P^2 S}$$

$$\psi \approx e^{im_P^2 S_0}$$

$$S = S_0 + m_P^{-2} S_1 + O(m_P^{-4})$$

$$\nabla S_0 \cdot \nabla S_0 + U = 0$$

If S_0 is real , the the wave function is oscillatory and peaked at

$$\Pi_{ij} = m_P^2 \frac{\delta S_0}{\delta h^{ij}}$$

$$\Pi_\phi = m_P^2 \frac{\delta S_0}{\delta \phi}$$

Stationary phase method for tomograms

$$w(X,\mu,\nu) = \frac{1}{2\pi|\nu|}\left|\int |\psi(y)| \exp\left(S(y) + \frac{i\mu}{2\nu}y^2 - \frac{iX}{\nu}y\right)dy\right|^2$$

The stationary point is determined by imposing the derivative of the phase to be 0

$$\frac{\partial S}{\partial q} + \frac{\mu}{\nu}q - \frac{X}{\nu} = 0$$

$$\frac{\partial S}{\partial q} = p$$

$$X = \mu q + \nu p$$

$$\hat{X} = \mu \hat{q} + \nu \hat{p}$$

Stationary phase method (2)

$$W(X,\mu,\nu) \approx |\psi(q_0)||(q_0)\nu + \mu|^{-1}$$

$$W(X,\mu,\nu) \approx \left| \sum_{i=1}^{N} \frac{\psi^i(q_0)}{\sqrt{\left|\frac{\partial^2 S_0}{\partial q^2}\right|_{q_0^i} \nu + \mu}} \right|^2$$

Noether symmetries for tomograms

$$| \psi >= \sum_{j=1}^{m} e^{i\Sigma_j Q^j} \; | \chi(Q^l) >$$

$$m < l \leq n$$

$$W(X_1,..., X_n, \mu_1,..., \mu_n, \nu_1,..., \nu_n) = \sum_{j=1}^{n} \frac{1}{\mu_j} \left| \int \chi\left(Q^i\right)\left(\prod_{l=m+1}^{n} e^{i(\mu/2\nu)(Q^i)^2 - i(X/\nu)(Q^i)} d(Q^i) \right) \right|^2$$

Noether Symmetries and Decoherence

- A good test to see verify the Hartle criterion is to see if the system decohere
- Decoherence can be calculated from the tomograms

$$\int W(X,\mu,\nu)W(Y,\mu,\nu)e^{i(Y-X)}dXdY \to 0$$

Work is in progress

Quantum and classical cosmological perturbations

- For the moment work is just beginning
- Classical field theory for gauge invariant perturbations and quantization
- Possibility to calculate the perturbation spectrum and the correlation function and compare with observations
- No interpretation problems

Conclusions……

- The tomographic approach gives is formulated in terms of a function that evolves with a classical-like equation
- It is useful for interpretation problems
- Comparison between classical variables and their quantum counterpart
- One can measure decoherence
- Analyze the perturbation spectrum

….and Perspectives

- Analyze quantum decoherence from our point of view (work in progress)
- Study the role of the quantum potential (work starting)
- Formulate a classical and quantum theory of fluctuations in cosmology (work starting)
- Formulate a classical and quantum theory of fluctuations in G.R.
- Determine how to measure a cosmological tomogram
- Extension of our analysis to Quantum Gravity

The symbol of an operator

Spazio di Hilbert H

Operatori $U(\vec{x})$ $\vec{x} = (x_1, \ldots\ldots, x_n)$

$\hat{A}$ $f_{\hat{A}}(\vec{x}) = Tr(\hat{A}\hat{U}(\vec{x}))$

Operatori $\hat{D}(\vec{x})$

$$\hat{A} = \int f_{\hat{A}}(\vec{x})\hat{D}(x)d\vec{x}$$

$$f_{\hat{A}}(\vec{x}) * f_{\hat{B}}(\vec{x}) = Tr(\hat{A}\hat{B}\hat{U})$$

Funzione di Wigner

$$\hat{U}(q,p) = 2\hat{D}(\alpha)(-1)^{a^+a}\hat{D}(-\alpha)$$

$$\hat{D}(q,p) = \frac{1}{\pi}\hat{D}(\alpha)(-1)^{a^+a}\hat{D}(-\alpha)$$

$$\hat{a} = \tfrac{1}{\sqrt{2}}\left(\hat{q} + i\hat{p}\right)$$

$$a = \tfrac{1}{\sqrt{2}}\left(q + ip\right)$$

$$D(\alpha) = \exp(\alpha\,\hat{a}^+ + \alpha^*\hat{a})$$

$$W_{\hat{A}} = 2Tr(\hat{A}\hat{D}(\alpha)(-1)^{\hat{a}^+\hat{a}}\hat{D}(-\alpha)$$

$\hat{A} = \hat{\rho}$ $\qquad W_{\hat{A}}$ è la funzione di Wigner

Tomographic symbol

$$\hat{U}(X,\mu,\nu) = \delta(X - \mu\hat{q} - \nu\hat{p})$$

$$D(X,\mu,\nu) = \tfrac{1}{2\pi}\exp(iX)\exp i(\mu\hat{q} + \nu\hat{p})$$

$$W_{\hat{A}}(X,\mu,\nu) = Tr(\hat{A}\delta(X - \mu\hat{q} - \nu\hat{p})$$

$$\hat{A} = \tfrac{1}{2\pi}\int W_{\hat{A}}(X,\mu,\nu)e^{i(X-\mu\hat{q}-\nu\hat{p})}dXd\mu\, d\nu$$